

Monoid Theory in Alonzo

A Little Theories Formalization in Simple Type Theory

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Abstract

Alonzo is a practice-oriented classical higher-order logic that extends first-order logic and that admits undefined expressions. Named in honor of Alonzo Church, Alonzo is based on Church’s type theory, Church’s formulation of simple type theory. The *little theories method* is a method for formalizing mathematical knowledge as a *theory graph* consisting of *theories* as nodes and *theory morphisms* as directed edges. The development of a mathematical topic is done in the “little theory” in the theory graph that has the most convenient level of abstraction and the most convenient vocabulary, and then the definitions and theorems produced in the development are transported, as needed, to other theories via the theory morphisms in the theory graph. The purpose of this paper is to illustrate how a body of mathematical knowledge can be formalized in Alonzo using the little theories method. This is done by formalizing *monoid theory* — the body of mathematical knowledge about monoids — in Alonzo.

Keywords: formal mathematics, simple type theory, little theories method, monoids, theory graphs, mathematical knowledge management.

MSC classification codes: 03B16, 03B38, 68V20, 68V30.

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1 Introduction

Formal mathematics is mathematics done within a formal logic. *Formalization* is the act of expressing mathematical knowledge in a formal logic. One of the chief benefits of formal mathematics is that a body of mathematical knowledge can be formalized as a precise, rigorous, and highly organized structure. This structure records the relationships between mathematical ideas and the contexts that govern mathematical results. Since it is based on a formal logic, it can be developed and analyzed using software.

An attractive and powerful method for organizing mathematical knowledge is the *little theories method* [21]. A body of mathematical knowledge is represented in the form of a *theory graph* [31] consisting of *theories* as nodes and *theory morphisms* as directed edges. Each mathematical topic is developed in the “little theory” in the theory graph that has the most convenient level of abstraction and the most convenient vocabulary. Then the definitions and theorems produced in the development are transported, as needed, from this abstract theory to other, usually more concrete, theories in the graph via the theory morphisms in the graph.

The purpose of this paper is to illustrate how a body of mathematical knowledge can be formalized in Alonzo [20], a practice-oriented classical higher-order logic that extends first-order logic, using the little theories method. Named in honor of Alonzo Church, Alonzo is based on Church’s type theory [8], Church’s formulation of simple type theory [17], and is closely related to Peter Andrews’ \mathcal{Q}_0 [1] and LUTINS [13, 14, 15], the logic of the IMPS proof assistant [22]. Unlike traditional predicate logics, Alonzo admits partial functions and undefined expressions in accordance with the approach employed in mathematical practice that we call the *traditional approach to undefinedness* [16]. Since partial functions naturally arise from theory morphisms [15], the little theories method works best with a logic like Alonzo that supports partial functions.

Alonzo has a simple syntax with a *formal notation* for machines and a *compact notation* for humans that closely resembles the notation found in mathematical practice. The compact notation is defined by the extensive set of *notational definitions and conventions* given in [20]. Alonzo has two semantics, one for mathematics based on *standard models* and one for logic based on Henkin-style *general models* [28]. By virtue of its syntax and semantics, Alonzo is exceptionally well suited for expressing and reasoning about mathematical ideas and, in particular, mathematical structures.

We have chosen *monoid theory* — the concepts, properties, and facts about monoids — as a sample body of mathematical knowledge to formalize

in Alonzo. A *monoid* is a mathematical structure consisting of a nonempty set, an associative binary function on the set, and a member of the set that is an identity element with respect to the function. Monoids are abundant in mathematics and computing. Single-object categories are monoids. Groups are monoids in which every element has an inverse. And several algebraic structures, such as rings, fields, Boolean algebras, and vector spaces, contain monoids as substructures.

Since a monoid is a significantly simpler algebraic structure than a group, monoid theory lacks the rich structure of group theory. We are formalizing monoid theory in Alonzo, instead of group theory, since it has just enough structure to adequately illustrate how a body of mathematical knowledge can be formalized in Alonzo. We will see that employing the little theories method in the formalization of monoid theory in Alonzo naturally leads to a robust theory graph.

Alonzo is equipped with a set of *mathematical knowledge modules* (*modules* for short) for constructing various kinds of mathematical knowledge units. For example, it has modules for constructing “theories” and “theory morphisms”. A *language* (or *signature*) of Alonzo is a pair $L = (\mathcal{B}, \mathcal{C})$, where \mathcal{B} is a finite set of base types and \mathcal{C} is a set of constants, that specifies a set of expressions. A *theory* of Alonzo is a pair $T = (L, \Gamma)$ where L is a language called the *language of T* and Γ is a set of sentences of L called the *axioms of T* . And a *theory morphism* from a theory T_1 to a theory T_2 is a mapping of the expressions of T_1 to the expressions of T_2 such that (1) base types are mapped to types and closed quasitypes (expressions that denote sets of values), (2) constants are mapped to closed expressions of appropriate type, and (3) valid sentences are mapped to valid sentences.

Alonzo also has modules for constructing “developments” and “development morphisms”. A *theory development* (or *development* for short) of Alonzo is a pair $D = (T, \Xi)$ where T is a theory and Ξ is a (possibly empty) sequence of definitions and theorems presented, respectively, as definition and theorem packages (see [20, Section 11.1]). T is called the *bottom theory* of D , and T' , the extension of T obtained by adding the definitions in Ξ to T , is called the *top theory* of D . We say that D is a *development of T* . A *development morphism* from a development D_1 to a development D_2 is a mapping from the expressions of D_1 to the expressions of D_2 that restricts to a theory morphism from the bottom theory of D_1 to the bottom theory of D_2 and that uniquely extends to a theory morphism from the top theory of D_1 to the top theory of D_2 (see [20, Section 13.4.1]). Theories and theory morphisms are special cases of developments and development morphisms, respectively, since we identify a theory T with the trivial development $(T, [])$.

The modules for constructing developments and development morphisms provide the means to represent knowledge in the form of a *development graph*, a richer kind of theory graph, in which the nodes are developments and the directed edges are development morphisms. Alonzo includes modules for transporting definitions and theorems from one development to another via development morphisms. The design of Alonzo’s module system is inspired by the IMPS implementation of the little theories method [21, 22].

Although [20] presents a simple and elegant proof system for Alonzo which is derived from Peter Andrews’ proof system for \mathcal{Q}_0 [1], in this paper we validate the definitions and theorems in a development using traditional mathematical proof. In this paper, the proofs are not included in the modules used to construct developments. Instead, they are given separately in Appendix A. Alonzo has not been implemented as a proof assistant. However, since Alonzo is closely related to LUTINS [13, 14, 15], the logic of the IMPS proof assistant [22], Alonzo can be implemented in much the same way that LUTINS is implemented in IMPS.

We produced the formalization of monoid theory with just a minimal amount of software support. We used the set of LaTeX macros and environments for Alonzo given in [19] plus a few macros created specifically for this paper. The macros are for presenting Alonzo types and expressions in both the formal and compact notations. The environments are for presenting Alonzo mathematical knowledge modules. The Alonzo modules are printed in brown color.

Alonzo is fully presented in [20]. Due to space limitations, we cannot duplicate the presentation of Alonzo in this paper. So we assume that the reader is familiar with [20]. The name of a (nontrivial) development of a theory named X will be given a name of the form X - n where n is a positive integer. The notational definitions that we need for our formalization of monoids that are not found in [20] are given in Tables 1–3. Table 1 defines new special notation. Table 2 defines several parametric pseudoconstants that serve as polymorphic logical constants (see [20, Section 6.1]). And Table 3 defines several useful abbreviations. The notational definitions introduced in this table have the form

$$A(\mathbf{B}_{\alpha_1}^1, \dots, \mathbf{B}_{\alpha_n}^n) \text{ stands for } C$$

where A is a name written in uppercase, $n \geq 0$, the syntactic variables $\mathbf{B}_{\alpha_1}^1, \dots, \mathbf{B}_{\alpha_n}^n$ appear in the expression C , and the bound variables introduced in C are chosen so that they are not free in $\mathbf{B}_{\alpha_1}^1, \dots, \mathbf{B}_{\alpha_n}^n$. For example, the bound variable $(x : \alpha)$ in the RHS of the first notational definition in Table 3 is chosen so that it is not free in $\mathbf{M}_{\{\alpha\}}$, $\mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}$, and \mathbf{E}_α .

| | | |
|--|------------|--|
| $\mathcal{P}(\mathbf{Q}_{\{\alpha\}})$ | stands for | $\{s : \{\alpha\} \mid s \subseteq \mathbf{Q}_{\{\alpha\}}\}$. |
| $\left(\prod_{i=\mathbf{M}_R}^{\mathbf{N}_R} \mathbf{A}_M \right)$ | stands for | $\text{prod}_{R \rightarrow R \rightarrow (R \rightarrow M) \rightarrow M} \mathbf{M}_R \mathbf{N}_R (\lambda i : R . \mathbf{A}_M)$. |
| $(\mathbf{X}_{R \rightarrow A} \mathbf{Y}_{R \rightarrow A})$ | stands for | $\mathbf{X}_{R \rightarrow A} \text{cat } \mathbf{Y}_{R \rightarrow A}$. |
| $(\mathbf{S}_{\{R \rightarrow A\}} \mathbf{T}_{\{R \rightarrow A\}})$ | stands for | $\mathbf{S}_{\{R \rightarrow A\}} \text{set-cat } \mathbf{T}_{\{R \rightarrow A\}}$. |
| $\left(\text{cat}_{i=\mathbf{M}_R}^{\mathbf{N}_R} \mathbf{A}_{R \rightarrow A} \right)$ | stands for | $\text{iter-cat}_{R \rightarrow R \rightarrow (R \rightarrow (R \rightarrow A)) \rightarrow (R \rightarrow A)} \mathbf{M}_R \mathbf{N}_R (\lambda i : R . \mathbf{A}_{R \rightarrow A})$. |

Table 1: Notational Definitions for Monoids: Special Notation

| |
|---|
| $\text{set-op}_{((\alpha \times \beta) \rightarrow \gamma) \rightarrow ((\{\alpha\} \times \{\beta\}) \rightarrow \{\gamma\})}$ |
| stands for |
| $\lambda f : (\alpha \times \beta) \rightarrow \gamma . \lambda p : \{\alpha\} \times \{\beta\} .$ $\{z : \gamma \mid \exists x : \text{fst } p, y : \text{snd } p . z = f(x, y)\}$. |
| $\text{id}_{\alpha \rightarrow \alpha}$ |
| stands for |
| $\lambda x : \alpha . x$. |
| $\circ_{((\alpha \rightarrow \beta) \times (\beta \rightarrow \gamma)) \rightarrow (\alpha \rightarrow \gamma)}$ |
| stands for |
| $\lambda p : (\alpha \rightarrow \beta) \times (\beta \rightarrow \gamma) . \lambda x : \alpha . (\text{snd } p) ((\text{fst } p) x)$. |
| $(\mathbf{F}_{\alpha \rightarrow \beta} \circ \mathbf{G}_{\beta \rightarrow \gamma})$ |
| stands for |
| $\circ_{((\alpha \rightarrow \beta) \times (\beta \rightarrow \gamma)) \rightarrow (\alpha \rightarrow \gamma)} (\mathbf{F}_{\alpha \rightarrow \beta}, \mathbf{G}_{\beta \rightarrow \gamma})$. |
| $\bullet_{((\alpha \rightarrow \beta) \times \alpha) \rightarrow \beta}$ |
| stands for |
| $\lambda p : (\alpha \rightarrow \beta) \times \alpha . (\text{fst } p) (\text{snd } p)$. |

Table 2: Notational Definitions for Monoids: Pseudoconstants

| |
|--|
| <p>MONOID($\mathbf{M}_{\{\alpha\}}, \mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}, \mathbf{E}_\alpha$)</p> <p>stands for</p> $\mathbf{M}_{\{\alpha\}} \downarrow \wedge$ $\mathbf{M}_{\{\alpha\}} \neq \emptyset_{\{\alpha\}} \wedge$ $\mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha} \downarrow (\mathbf{M}_{\{\alpha\}} \times \mathbf{M}_{\{\alpha\}}) \rightarrow \mathbf{M}_{\{\alpha\}} \wedge$ $\mathbf{E}_\alpha \downarrow \mathbf{M}_{\{\alpha\}} \wedge$ $\forall x, y, z : \mathbf{M}_{\{\alpha\}} \cdot$ $\mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha} (x, \mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha} (y, z)) = \mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha} (\mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha} (x, y), z) \wedge$ $\forall x : \mathbf{M}_{\{\alpha\}} \cdot \mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha} (\mathbf{E}_\alpha, x) = \mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha} (x, \mathbf{E}_\alpha) = x.$ |
| <p>COM-MONOID($\mathbf{M}_{\{\alpha\}}, \mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}, \mathbf{E}_\alpha$)</p> <p>stands for</p> $\text{MONOID}(\mathbf{M}_{\{\alpha\}}, \mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}, \mathbf{E}_\alpha) \wedge$ $\forall x, y : \mathbf{M}_{\{\alpha\}} \cdot \mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha} (x, y) = \mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha} (y, x)$ |
| <p>MON-ACTION($\mathbf{M}_{\{\alpha\}}, \mathbf{S}_{\{\beta\}}, \mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}, \mathbf{E}_\alpha, \mathbf{G}_{(\alpha \times \beta) \rightarrow \beta}$)</p> <p>stands for</p> $\text{MONOID}(\mathbf{M}_{\{\alpha\}}, \mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}, \mathbf{E}_\alpha) \wedge$ $\mathbf{S}_{\{\beta\}} \downarrow \wedge$ $\mathbf{S}_{\{\beta\}} \neq \emptyset_{\{\beta\}} \wedge$ $\mathbf{G}_{(\alpha \times \beta) \rightarrow \beta} \downarrow (\mathbf{M}_{\{\alpha\}} \times \mathbf{S}_{\{\beta\}}) \rightarrow \mathbf{S}_{\{\beta\}} \wedge$ $\forall x, y : \mathbf{M}_{\{\alpha\}}, s : \mathbf{S}_{\{\beta\}} \cdot$ $\mathbf{G}_{(\alpha \times \beta) \rightarrow \beta} (x, \mathbf{G}_{(\alpha \times \beta) \rightarrow \beta} (y, s)) = \mathbf{G}_{(\alpha \times \beta) \rightarrow \beta} (\mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha} (x, y), s) \wedge$ $\forall s : \mathbf{S}_{\{\beta\}} \cdot \mathbf{G}_{(\alpha \times \beta) \rightarrow \beta} (\mathbf{E}_\alpha, s) = s.$ |
| <p>MON-HOMOM($\mathbf{M}_{\{\alpha\}}^1, \mathbf{M}_{\{\beta\}}^2, \mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}^1, \mathbf{E}_\alpha^1, \mathbf{F}_{(\beta \times \beta) \rightarrow \beta}^2, \mathbf{E}_\beta^2, \mathbf{H}_{\alpha \rightarrow \beta}$)</p> <p>stands for</p> $\text{MONOID}(\mathbf{M}_{\{\alpha\}}^1, \mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}^1, \mathbf{E}_\alpha^1) \wedge$ $\text{MONOID}(\mathbf{M}_{\{\beta\}}^2, \mathbf{F}_{(\beta \times \beta) \rightarrow \beta}^2, \mathbf{E}_\beta^2) \wedge$ $\mathbf{H}_{\alpha \rightarrow \beta} \downarrow \mathbf{M}_{\{\alpha\}}^1 \rightarrow \mathbf{M}_{\{\beta\}}^2 \wedge$ $\forall x, y : \mathbf{M}_{\{\alpha\}}^1 \cdot \mathbf{H}_{\alpha \rightarrow \beta} (\mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}^1 (x, y)) = \mathbf{F}_{(\beta \times \beta) \rightarrow \beta}^2 (\mathbf{H}_{\alpha \rightarrow \beta} x, \mathbf{H}_{\alpha \rightarrow \beta} y) \wedge$ $\mathbf{H}_{\alpha \rightarrow \beta} \mathbf{E}_\alpha^1 = \mathbf{E}_\beta^2$ |

Table 3: Notational Definitions for Monoids: Abbreviations

The standard approach to formal mathematics emphasizes *certification*: (1) mathematics is done with the help of a proof assistant and (2) all details are formally proved and mechanically checked. The formalization of monoid theory presented in this paper exemplifies an alternative approach to formal mathematics [18] that emphasizes *communication*: (1) proofs are written in a traditional (nonformal) style, (2) the underlying logic is designed to be as close to mathematical practice as possible, (3) mathematical knowledge is organized as a development graph using the little theories method, and (4) supporting software systems do not need a formal proof system and thus are much easier to build and use than proof assistants. The result is an approach in which everything is done within a formal logic except for proofs and the entire development is optimized for communication. We will call the standard approach the *certification-oriented approach* and the alternative approach the *communication-oriented approach*.

The paper is organized as follows. Sections 2–9 present developments of theories of monoids, commutative monoids, transformation monoids, monoid actions, monoid homomorphisms, and monoids over real number arithmetic, respectively, plus some supporting developments. These developments have been constructed to be illustrative; they are not intended to be complete in any sense. Sections 2–9 also present various development morphisms that are used to transport definitions and theorems from one development to another. Section 10 shows how our formalization of monoid theory can support a theory of strings. Related work is discussed in Section 11. The paper concludes in Section 12 with a summary and some final remarks. The definitions and theorems of the developments we have constructed are validated by traditional mathematical proofs presented in Appendix A. Appendix B contains some miscellaneous theorems needed for the proofs in Appendix A.

2 Monoids

A *monoid* is a mathematical structure (m, \cdot, e) where m is a nonempty set of values, $\cdot : (m \times m) \rightarrow m$ is an associative function, and $e \in m$ is an identity element with respect to \cdot . Mathematics and computing are replete with examples of monoids such as $(\mathbb{N}, +, 0)$, $(\mathbb{N}, *, 1)$, and $(\Sigma^*, ++, \epsilon)$ where Σ^* is the set of strings over an alphabet Σ , $++$ is string concatenation, and ϵ is the empty string.

Let $T = (L, \Gamma)$ be a theory of Alonzo. Consider a tuple

$$(\zeta_\alpha, \mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}, \mathbf{E}_\alpha)$$

where (1) ζ_α is either a type α of L or a closed quasitype $\mathbf{Q}_{\{\alpha\}}$ of L and (2) $\mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}$ and \mathbf{E}_α are closed expressions of L . Let \mathbf{X}_o be the sentence

$$\text{MONOID}(\mathbf{M}_{\{\alpha\}}, \mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}, \mathbf{E}_\alpha),$$

where MONOID is the abbreviation introduced by the notational definition given in Table 3 and $\mathbf{M}_{\{\alpha\}}$ is $U_{\{\alpha\}}$ if ζ_α is α and is $\mathbf{Q}_{\{\alpha\}}$ otherwise. If $T \models \mathbf{X}_o$, then $(\zeta_\alpha, \mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}, \mathbf{E}_\alpha)$ denotes a monoid in T . Stated more precisely, if $T \models \mathbf{X}_o$, then, for all general models M of T and all assignments $\varphi \in \text{assign}(M)$, $(\zeta_\alpha, \mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}, \mathbf{E}_\alpha)$ denotes the monoid

$$(V_\varphi^M(\mathbf{M}_{\{\alpha\}}), V_\varphi^M(\mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}), V_\varphi^M(\mathbf{E}_\alpha)).$$

Thus we can show that $(\zeta_\alpha, \mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}, \mathbf{E}_\alpha)$ denotes a monoid in T by proving $T \models \mathbf{X}_o$. However, we may need general definitions and theorems about monoids to prove properties in T about the monoid denoted by $(\zeta_\alpha, \mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}, \mathbf{E}_\alpha)$. It would be extremely inefficient to state these definitions and prove these theorems in T since instances of these same definitions and theorems could easily be needed for other triples in T , as well as in other theories, that denote monoids.

Instead of developing part of a monoid theory in T , we should apply the little theories method and develop a “little theory” T_{mon} of monoids, separate from T , that has the most convenient level of abstraction and the most convenient vocabulary for talking about monoids. The general definitions and theorems of monoids can then be introduced in a development D_{mon} of T_{mon} in a universal abstract form. When these definitions and theorems are needed in a development D , a development morphism from D_{mon} to D can be created and then used to transport the abstract definitions and theorems in D_{mon} to concrete instances of them in D . The validity of these concrete definitions and theorems in D is guaranteed by the fact that the abstract definitions and theorems are valid in the top theory of D_{mon} and the development morphism used to transport them preserves validity.

We can verify that $(\zeta_\alpha, \mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}, \mathbf{E}_\alpha)$ denotes a monoid in T by simply constructing an appropriate theory morphism Φ from T_{mon} to T . As a bonus, we can use Φ to transport the abstract definitions and theorems in T_{mon} to concrete instances of them in T (or a development of T) whenever they are needed. Moreover, we do not have to explicitly prove that \mathbf{X}_o is valid in T ; instead, we only need to show that there is an abstract theorem of T_{mon} that Φ transports to \mathbf{X}_o .

The following theory definition module defines a suitably abstract theory of monoids named MON:

Theory Definition 2.1 (Monoids)

Name: MON.

Base types: M .

Constants: $\cdot_{(M \times M) \rightarrow M}$, e_M .

Axioms:

1. $\forall x, y, z : M . x \cdot (y \cdot z) = (x \cdot y) \cdot z$ (\cdot is associative).
2. $\forall x : M . e \cdot x = x \cdot e = x$ (e is an identity element with respect to \cdot).

Notice that we have employed several notational definitions and conventions in the axioms — including dropping the types of the constants — for the sake of brevity. This theory specifies the set of monoids exactly: The base type M , like all types, denotes a nonempty set m ; the constant $\cdot_{(M \times M) \rightarrow M}$ denotes a function $\cdot : (m \times m) \rightarrow m$ that is associative; and the constant e_m denotes a member e of m that is an identity element with respect to \cdot .

The following development definition module defines a development, named MON-1, of the theory MON:

Development Definition 2.2 (Monoids 1)

Name: MON-1.

Bottom theory: MON.

Definitions and theorems:

Thm1: $\text{MONOID}(U_{\{M\}}, \cdot_{(M \times M) \rightarrow M}, e_M)$
(models of MON define monoids).

Thm2: $\text{TOTAL}(\cdot_{(M \times M) \rightarrow M})$ (\cdot is total).

Thm3: $\forall x : M . (\forall y : M . x \cdot y = y \cdot x = y) \Rightarrow x = e$
(uniqueness of identity element).

Def1: $\text{submonoid}_{\{M\} \rightarrow o} =$
 $\lambda s : \{M\} . s \neq \emptyset_{\{M\}} \wedge (\cdot \upharpoonright_{s \times s} \downarrow (s \times s) \rightarrow s) \wedge e \in s$ (submonoid).

Thm4: $\forall s : \{M\} . \text{submonoid } s \Rightarrow \text{MONOID}(s, \cdot \upharpoonright_{s \times s}, e)$
(submonoids are monoids).

Thm5: $\text{submonoid } \{e\}$ (minimum submonoid).

Thm6: submonoid $U_{\{M\}}$ (maximum submonoid).

Def2: $\cdot_{(M \times M) \rightarrow M}^{\text{op}} = \lambda p : M \times M . (\text{snd } p) \cdot (\text{fst } p)$ (opposite of \cdot).

Thm7: $\forall x, y, z : M . x \cdot^{\text{op}} (y \cdot^{\text{op}} z) = (x \cdot^{\text{op}} y) \cdot^{\text{op}} z$
(\cdot^{op} is associative).

Thm8: $\forall x : M . e \cdot^{\text{op}} x = x \cdot^{\text{op}} e = x$
(e is an identity element with respect to \cdot^{op}).

Def3: $\odot_{(\{M\} \times \{M\}) \rightarrow \{M\}} = \text{set-op}_{((M \times M) \rightarrow M) \rightarrow ((\{M\} \times \{M\}) \rightarrow \{M\})} \cdot$
(set product).

Def4: $E_{\{M\}} = \{e_M\}$ (set identity element).

Thm9: $\forall x, y, z : \{M\} . x \odot (y \odot z) = (x \odot y) \odot z$ (\odot is associative).

Thm10: $\forall x : \{M\} . E \odot x = x \odot E = x$
(E is an identity element with respect to \odot).

$\text{set-op}_{((M \times M) \rightarrow M) \rightarrow ((\{M\} \times \{M\}) \rightarrow \{M\})}$ is an instance of the parametric pseudoconstant $\text{set-op}_{((\alpha \times \beta) \rightarrow \gamma) \rightarrow ((\{\alpha\} \times \{\beta\}) \rightarrow \{\gamma\})}$ defined in Table 2.

Thm1 states that each model of MON defines a monoid. Thm2 states the monoid's binary function is total. Thm3 states that a monoid's identity element is unique. Def1 defines the notion of a *submonoid* and Thm4–Thm6 are three theorems about submonoids. Notice that $\cdot|_{s \times s}$, the restriction of \cdot to $s \times s$, denotes a partial function. Notice also that

$$\cdot|_{s \times s} \downarrow (s \times s) \rightarrow s$$

in Def1 asserts that s is closed under $\cdot|_{s \times s}$ since \cdot is total by Thm2. Def2 defines $\cdot_{(M \times M) \rightarrow M}^{\text{op}}$, the *opposite* of \cdot , and Thm7–Thm8 are key theorems about \cdot^{op} . Def3 defines $\odot_{(\{M\} \times \{M\}) \rightarrow \{M\}}$, the *set product* on $\{M\}$; Def4 defines $E_{\{M\}}$, the identity element with respect to \odot ; and Thm9–Thm10 are key theorems about \odot . These four definitions and ten theorems require proofs that show the RHS of each definition (i.e., the definition's definiens) is defined and each theorem is valid. The proofs are given in Appendix A.

3 Transportation of Definitions and Theorems

Let T be a theory such that $T \models \mathbf{X}_o$ where \mathbf{X}_o is

$$\text{MONOID}(\mathbf{M}_{\{\alpha\}}, \mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}, \mathbf{E}_{\alpha}),$$

and assume that D is some development of T (which could be T itself). We would like to show how the definitions and theorems of the development MON-1 can be transported to D .

Before considering the general case, we will consider the special case when $\mathbf{M}_{\{\alpha\}}$ is $U_{\{\alpha\}}$, which denotes the entire domain for the type α , and $\mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}$ and \mathbf{E}_α are constants $\mathbf{c}_{(\alpha \times \alpha) \rightarrow \alpha}$ and \mathbf{d}_α . We start by defining a theory morphism from MON to T using a theory translation definition module:

Theory Translation Definition 3.1 (Special MON to T)

Name: special-MON-to- T .

Source theory: MON.

Target theory: T .

Base type mapping:

1. $M \mapsto \alpha$.

Constant mapping:

1. $\cdot_{(M \times M) \rightarrow M} \mapsto \mathbf{c}_{(\alpha \times \alpha) \rightarrow \alpha}$.
2. $e_M \mapsto \mathbf{d}_\alpha$.

Since special-MON-to- T is a normal translation, it has no obligations of the first kind by [20, Lemma 13.10] and two obligations of the second kind which are valid in T by [20, Lemma 13.11]. It has two obligations of the third kind corresponding to the two axioms of MON. $T \models \mathbf{X}_o$ implies that each of these two obligations is valid in T . Therefore, special-MON-to- T is a theory morphism from MON to T by the Morphism Theorem [20, Theorem 13.13].

Now we can transport the definitions and theorems of MON-1 to D via special-MON-to- T using definition and theorems transportation modules.¹ For example, Thm3 and Def1 can be transported using the following two modules:

¹These transportation modules include “source development” and “target development” fields that are missing in the transportation modules given in [20].

Theorem Transportation 3.2 (Transport of Thm3 to D)

Name: uniqueness-of-identity-element-via-special-MON-to- D .

Source development: MON-1.

Target development: D .

Development morphism: special-MON-to- T .

Theorem:

$$\text{Thm3: } \forall x : M . (\forall y : M . x \cdot y = y \cdot x = y) \Rightarrow x = \mathbf{e} \\ \text{(uniqueness of identity element).}$$

Transported theorem:

$$\text{Thm3-via-special-MON-to-}T: \\ \forall x : \alpha . (\forall y : \alpha . x \mathbf{c} y = y \mathbf{c} x = y) \Rightarrow x = \mathbf{d} \\ \text{(uniqueness of identity element).}$$

New target development: D' .

Definition Transportation 3.3 (Transport of Def1 to D')

Name: submonoid-via-special-MON-to- D' .

Source development: MON-1.

Target development: D' .

Development morphism: special-MON-to- T .

Definition:

$$\text{Def1: submonoid}_{\{M\} \rightarrow o} = \\ \lambda s : \{M\} . s \neq \emptyset_{\{M\}} \wedge (\cdot \upharpoonright_{s \times s} \downarrow (s \times s) \rightarrow s) \wedge \mathbf{e} \in s \quad \text{(submonoid).}$$

Transported definition:

$$\text{Def1-via-special-MON-to-}T: \text{submonoid}_{\{\alpha\} \rightarrow o} = \\ \lambda s : \{\alpha\} . s \neq \emptyset_{\{\alpha\}} \wedge (\mathbf{c} \upharpoonright_{s \times s} \downarrow (s \times s) \rightarrow s) \wedge \mathbf{d} \in s \quad \text{(submonoid).}$$

New target development: D'' .

New development morphism: special-MON-1-to- D'' .

We will next consider the general case when $\mathbf{M}_{\{\alpha\}}$ may be different from $U_{\{\alpha\}}$ and $\mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}$ and \mathbf{E}_α may not be constants. The general case is usually more complicated and less succinct than the special case. We start again by defining a theory morphism from MON to T using a theory translation definition module:

Theory Translation Definition 3.4 (General MON to T)

Name: general-MON-to- T .

Source theory: MON.

Target theory: T .

Base type mapping:

1. $M \mapsto \mathbf{M}_{\{\alpha\}}$.

Constant mapping:

1. $\cdot_{(M \times M) \rightarrow M} \mapsto \mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}$.
2. $e_M \mapsto \mathbf{E}_\alpha$.

Let $\text{general-MON-to-}T = (\mu, \nu)$. Then $\text{general-MON-to-}T$ has the following five obligations (one of the first, two of the second, and two of the third kind):

1. $\bar{\nu}(U_{\{M\}} \neq \emptyset_{\{M\}}) \equiv (\lambda x : \mathbf{M}_{\{\alpha\}} . T_o) \neq (\lambda x : \mathbf{M}_{\{\alpha\}} . F_o)$.
2. $\bar{\nu}(\cdot_{(M \times M) \rightarrow M} \downarrow U_{\{(M \times M) \rightarrow M\}}) \equiv$
 $\mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha} \downarrow (\lambda x : (\mathbf{M}_{\{\alpha\}} \times \mathbf{M}_{\{\alpha\}}) \rightarrow \mathbf{M}_{\{\alpha\}} . T_o)$.
3. $\bar{\nu}(e_M \downarrow U_{\{M\}}) \equiv \mathbf{E}_\alpha \downarrow (\lambda x : \mathbf{M}_{\{\alpha\}} . T_o)$.
4. $\bar{\nu}(\forall x, y, z : M . x \cdot (y \cdot z) = (x \cdot y) \cdot z) \equiv$
 $\forall x, y, z : \mathbf{M}_{\{\alpha\}} .$
 $\mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}(x, \mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}(y, z)) = \mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}(\mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}(x, y), z)$.
5. $\bar{\nu}(\forall x : M . e \cdot x = x \cdot e = x) \equiv$
 $\forall x : \mathbf{M}_{\{\alpha\}} . \mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}(\mathbf{E}_\alpha, x) = \mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}(x, \mathbf{E}_\alpha) = x$.

$\mathbf{A}_\alpha \equiv \mathbf{B}_\alpha$ means the expressions denoted by \mathbf{A}_α and \mathbf{B}_α are identical.

$T \vDash \mathbf{X}_o$ implies that each of these obligations is valid in T as follows. The first and second conjuncts of \mathbf{X}_o imply that the first obligation is valid in T by part 3 of Lemma B.2. The first and third conjuncts imply that the second obligation is valid in T by part 5 of Lemma B.2. The first and fourth conjuncts imply that the third obligation is valid in T by part 5 of Lemma B.2. And the fifth and sixth conjuncts imply, respectively, that the fourth and fifth obligations are valid in T . Therefore, **general-MON-to- T** is a theory morphism by the Morphism Theorem [20, Theorem 13.13].

We can now transport, as before, the definitions and theorems of MON-1 to D via **general-MON-to- T** using definition and theorem transportation modules, but we can also transport them using a group transportation module². For example, Thm3 and Def1 can be transported as a group using the following group transportation module:

Group Transportation 3.5 (Transport of Thm3 and Def1 to D)

Name: uniqueness-of-identity-element-and-submonoid-to- D .

Source development: MON-1.

Target development: D .

Development morphism: general-MON-to- T .

Definitions and theorems:

Thm3: $\forall x : M . (\forall y : M . x \cdot y = y \cdot x = y) \Rightarrow x = e$
(uniqueness of identity element).

Def1: $\text{submonoid}_{\{M\} \rightarrow o} =$
 $\lambda s : \{M\} . s \neq \emptyset_{\{M\}} \wedge (\cdot \upharpoonright_{s \times s} \downarrow (s \times s) \rightarrow s) \wedge e \in s$ (submonoid).

Transported definitions and theorems:

Thm3-via-general-MON-to- T :

$\forall x : \mathbf{M}_{\{\alpha\}} .$
 $(\forall y : \mathbf{M}_{\{\alpha\}} . \mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}(x, y) = \mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}(y, x) = y) \Rightarrow x = \mathbf{E}_\alpha$
(uniqueness of identity element).

²This kind of module, which is not given in [20], transports a set of definitions and theorems in parallel.

$$\begin{aligned}
& \text{Def1-via-general-MON-to-}T: \text{ submonoid}_{\{\alpha\} \rightarrow o} = \\
& \lambda s : \mathcal{P}(\mathbf{M}_{\{\alpha\}}) . \\
& \quad s \neq (\lambda x : \mathbf{M}_{\{\alpha\}} . F_o) \wedge \\
& \quad (\mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha} \uparrow_{s \times s} \downarrow (s \times s) \rightarrow s) \wedge \\
& \quad \mathbf{E}_\alpha \in s \qquad \qquad \qquad \text{(submonoid)}.
\end{aligned}$$

New target development: D' .

New development morphism: general-MON-1-to- D' .

The abbreviation $\mathcal{P}(\mathbf{M}_{\{\alpha\}})$, which denotes the power set of $\mathbf{M}_{\{\alpha\}}$, is defined in Table 1.

4 Opposite and Set Monoids

For every monoid (m, \cdot, e) , there is (1) an associated monoid $(m, \cdot^{\text{op}}, e)$, where \cdot^{op} is the opposite of \cdot , called the *opposite monoid* of (m, \cdot, e) and (2) a monoid $(\mathcal{P}(m), \odot, \{e\})$, where $\mathcal{P}(m)$ is the power set of m and \odot is the set product on $\mathcal{P}(m)$, called the *set monoid* of (m, \cdot, e) .

We will construct a development morphism named MON-to-opposite-monoid from the theory MON to its development MON-1 that maps

$$(M, \cdot_{(M \times M) \rightarrow M}, \mathbf{e})$$

to

$$(M, \cdot_{(M \times M) \rightarrow M}^{\text{op}}, \mathbf{e}).$$

Then we will be able to use this morphism to transport abstract definitions and theorems about monoids to more concrete definitions and theorems about opposite monoids. Here is the definition of MON-to-opposite-monoid:

Development Translation Definition 4.1 (MON to Op. Monoid)

Name: MON-to-opposite-monoid.

Source development: MON.

Target development: MON-1.

Base type mapping:

1. $M \mapsto M$.

Constant mapping:

1. $\cdot_{(M \times M) \rightarrow M} \mapsto \cdot_{(M \times M) \rightarrow M}^{\text{op}}$
2. $e_M \mapsto e_M$.

Since MON-to-opposite-monoid is a normal translation, it has no obligations of the first kind by [20, Lemma 13.10] and two obligations of the second kind which are valid in the top theory of MON-1 by [20, Lemma 13.11]. It has two obligations of the third kind corresponding to the two axioms of MON. These two obligations are logically equivalent to Thm7 and Thm8, respectively, and these two theorems are obviously valid in the top theory of MON-1. Therefore, MON-to-opposite-monoid is a development morphism from MON to MON-1 by the Morphism Theorem [20, Theorem 13.13].

We can now transport Thm1 via MON-to-opposite-monoid to show that opposite monoids are indeed monoids:

Theorem Transportation 4.2 (Transport of Thm1 to MON-1)

Name: monoid-via-MON-to-opposite-monoid.

Source development: MON.

Target development: MON-1.

Development morphism: MON-to-opposite-monoid.

Theorem:

Thm1: $\text{MONOID}(U_{\{M\}}, \cdot_{(M \times M) \rightarrow M}, e_M)$
 (models of MON define monoids).

Transported theorem:

Thm11 (Thm1-via-MON-to-opposite-monoid):
 $\text{MONOID}(U_{\{M\}}, \cdot_{(M \times M) \rightarrow M}^{\text{op}}, e_M)$ (opposite monoids are monoids).

New target development: MON-2.

Similarly, we will construct a development morphism named MON-to-set-monoid from the theory MON to its development MON-2 that maps

$(M, \cdot_{(M \times M) \rightarrow M}, e_M)$

to

$$(\{M\}, \odot_{(\{M\} \times \{M\}) \rightarrow \{M\}}, E_{\{M\}}).$$

Then we will be able to use this morphism to transport abstract definitions and theorems about monoids to more concrete definitions and theorems about set monoids. Here is the definition of MON-to-set-monoid:

Development Translation Definition 4.3 (MON to Set Monoid)

Name: MON-to-set-monoid.

Source development: MON.

Target development: MON-2.

Base type mapping:

1. $M \mapsto \{M\}$.

Constant mapping:

1. $\cdot_{(M \times M) \rightarrow M} \mapsto \odot_{(\{M\} \times \{M\}) \rightarrow \{M\}}$.
2. $e_M \mapsto E_{\{M\}}$.

Since MON-to-set-monoid is a normal translation, it has no obligations of the first kind by [20, Lemma 13.10]. It has two obligations of the second kind. The first one is valid in the top theory of MON-2 by part 4 of Lemma B.2 since $\odot_{(\{M\} \times \{M\}) \rightarrow \{M\}}$ beta-reduces by [20, Axiom A4] to a function abstraction which is defined by [20, Axiom A5.11]. The second one is valid in the top theory of MON-2 by part 4 of Lemma B.2 since $E_{\{M\}}$ is a function abstraction which is defined by [20, Axiom A5.11]. It has two obligations of the third kind corresponding to the two axioms of MON. These two obligations are Thm9 and Thm10, respectively. and these two theorems are obviously valid in the top theory of MON-2. Therefore, MON-to-set-monoid is a development morphism from MON to MON-2 by the Morphism Theorem [20, Theorem 13.13].

We can now transport Thm1 via MON-to-set-monoid to show that set monoids are indeed monoids:

Theorem Transportation 4.4 (Transport of Thm1 to MON-2)

Name: monoid-via-MON-to-set-monoid.

Source development: MON.

Target development: MON-2.

Development morphism: MON-to-set-monoid.

Theorem:

Thm1: $\text{MONOID}(U_{\{M\}}, \cdot_{(M \times M) \rightarrow M}, e_M)$
(models of MON define monoids).

Transported theorem:

Thm12 (Thm1-via-MON-to-set-monoid):
 $\text{MONOID}(U_{\{\{M\}\}}, \odot_{(\{M\} \times \{M\}) \rightarrow \{M\}}, E_{\{M\}})$
(set monoids are monoids).

New target development: MON-3.

5 Commutative Monoids

A monoid (m, \cdot, e) is *commutative* if \cdot is commutative.

Let \mathbf{Y}_o be the formula

$$\text{COM-MONOID}(\mathbf{M}_{\{\alpha\}}, \mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}, \mathbf{E}_\alpha),$$

where COM-MONOID is the abbreviation introduced by the notational definition given in Table 3. \mathbf{Y}_o asserts that (m, \cdot, e) is a commutative monoid where m is a set denoting $\mathbf{M}_{\{\alpha\}}$, $\cdot : (m \times m) \rightarrow m$ is a function denoting $\mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}$, and e is an element of m denoting \mathbf{E}_α .

We can define a theory of commutative monoids, named COM-MON, by adding an axiom that says \cdot is commutative to the theory MON using a theory extension module:

Theory Extension 5.1 (Commutative Monoids)

Name: COM-MON.

Extends MON.

New base types:

New constants:

New axioms:

$$3. \forall x, y : M . x \cdot y = y \cdot x \quad (\cdot \text{ is commutative}).$$

Then we can develop the theory COM-MON using the following development definition module:

Development Definition 5.2 (Commutative Monoids 1)

Name: COM-MON-1.

Bottom theory: COM-MON.

Definitions and theorems:

Thm13: COM-MONOID($U_{\{M\}}, \cdot_{(M \times M) \rightarrow M}, e_M$)
(models of COM-MON define commutative monoids).

Def5: $\leq_{M \rightarrow M \rightarrow o} = \lambda x, y : M . \exists z : M . x \cdot z = y$ (weak order).

Thm14: $\forall x : M . x \leq x$ (reflexivity).

Thm15: $\forall x, y, z : M . (x \leq y \wedge y \leq z) \Rightarrow x \leq z$ (transitivity).

Thm13 states that each model of COM-MON defines a commutative monoid. Def5 defines a weak (nonstrict) order that is a pre-order by Thm14 and Thm15. We could have put Def5, Thm14, and Thm15 in a development of MON since Thm14 and Thm15 do not require that \cdot is commutative, but we have put these in COM-MON instead since $\leq_{M \rightarrow M \rightarrow o}$ is more natural for commutative monoids than for noncommutative monoids.

Since COM-MON is an extension of MON, there is an inclusion (i.e., a theory morphism whose mapping is the identity function) from MON to COM-MON. This inclusion is defined by the following theory translation definition module:

Theory Translation Definition 5.3 (MON to COM-MON)

Name: MON-to-COM-MON.

Source theory: MON.

Target theory: COM-MON.

Base type mapping:

1. $M \mapsto M$.

Constant mapping:

1. $\cdot_{(M \times M) \rightarrow M} \mapsto \cdot_{(M \times M) \rightarrow M}$.
2. $e_M \mapsto e_M$.

MON-to-COM-MON is a theory morphism from MON to COM-MON and thus is a development morphism from MON-3 to COM-MON-1. By virtue of this development morphism, the definitions and theorems of MON-3 can be freely transported verbatim to COM-MON-1. In the rest of the paper, when a theory T' is an extension of a theory T , D is a development of T , and D' is a development of T' , we will assume that the inclusion from T to T' is already defined and that the definitions and theorems of D are also definitions and theorems of D' .

6 Transformation Monoids

A very important type of monoid is a monoid composed of transformations of a set. Let s be a nonempty set. Then (f, \circ, id) , where f is a set of (partial or total) functions from s to s ,

$$\circ : ((s \rightarrow s) \times (s \rightarrow s)) \rightarrow (s \rightarrow s)$$

is function composition, and $\text{id} : s \rightarrow s$ is the identity function, is a *transformation monoid on s* if f is closed under \circ and $\text{id} \in f$. It is easy to verify that every transformation monoid is a monoid. If f contains every function in the function space $s \rightarrow s$, then (f, \circ, id) is clearly a transformation monoid which is called the *full transformation monoid on s* . Let us say that a transformation monoid (f, \circ, id) is *standard* if f contains only total functions. In many developments, nonstandard transformation monoids are ignored, but there is no reason to do that here since Alonzo admits undefined expressions and partial functions.

Consider the following theory ONE-BT of one base type:

Theory Definition 6.1 (One Base Type)

Name: ONE-BT.

Base types: S .

Constants:

Axioms:

This theory is sufficient for defining the notion of a transformation monoid in a development of it, but we must first introduce some general facts about function composition. To do that, we need a theory FUN-COMP with four base types in order to state the associativity theorem for function composition in full generality:

Theory Definition 6.2 (Function Composition)

Name: FUN-COMP.

Base types: A, B, C, D .

Constants:

Axioms:

We introduce three theorems for function composition in a development of FUN-COMP:

Development Definition 6.3 (Function Composition 1)

Name: FUN-COMP-1.

Bottom theory: FUN-COMP.

Definitions and theorems:

Thm16: $\forall f : A \rightarrow B, g : B \rightarrow C, h : C \rightarrow D . f \circ (g \circ h) = (f \circ g) \circ h$
(\circ is associative).

Thm17: $\forall f : A \rightarrow B . \text{id}_{A \rightarrow A} \circ f = f \circ \text{id}_{B \rightarrow B} = f$
(identity functions are left and right identity elements).

The parametric pseudoconstants $\circ_{((\alpha \rightarrow \beta) \times (\beta \rightarrow \gamma)) \rightarrow (\alpha \rightarrow \gamma)}$ and $\text{id}_{\alpha \rightarrow \alpha}$ are defined in Table 2. The infix notation for the application of

$$\circ_{((\alpha \rightarrow \beta) \times (\beta \rightarrow \gamma)) \rightarrow (\alpha \rightarrow \gamma)}$$

is also defined in Table 2.

Next we define a theory morphism from FUN-COMP to ONE-BT:

Theory Translation Definition 6.4 (FUN-COMP to ONE-BT)

Name: FUN-COMP-to-ONE-BT.

Source theory: FUN-COMP.

Target theory: ONE-BT.

Base type mapping:

1. $A \mapsto S$.
2. $B \mapsto S$.
3. $C \mapsto S$.
4. $D \mapsto S$.

Constant mapping:

The translation FUN-COMP-to-ONE-BT is clearly a theory morphism by the Morphism Theorem [20, Theorem 13.13] since it is a normal translation and FUN-COMP contains no constants or axioms. So we can transport the theorems of FUN-COMP-1 to ONE-BT via FUN-COMP-to-ONE-BT:

Group Transportation 6.5 (Transport of Thm16–Thm17 to ONE-BT)

Name: function-composition-theorems-via-FUN-COMP-to-ONE-BT.

Source development: FUN-COMP-1.

Target development: ONE-BT.

Development morphism: FUN-COMP-to-ONE-BT.

Definitions and theorems:

Thm16: $\forall f : A \rightarrow B, g : B \rightarrow C, h : C \rightarrow D . f \circ (g \circ h) = (f \circ g) \circ h$
(\circ is associative).

Thm17: $\forall f : A \rightarrow B . \text{id}_{A \rightarrow A} \circ f = f \circ \text{id}_{B \rightarrow B} = f$
(identity functions are left and right identity elements).

Transported definitions and theorems:

Thm18 (Thm16-via-FUN-COMP-to-ONE-BT):
 $\forall f, g, h : S \rightarrow S . f \circ (g \circ h) = (f \circ g) \circ h$ (\circ is associative).

Thm19 (Thm17-via-FUN-COMP-to-ONE-BT):
 $\forall f : S \rightarrow S . \text{id}_{S \rightarrow S} \circ f = f \circ \text{id}_{S \rightarrow S} = f$
 ($\text{id}_{S \rightarrow S}$ is an identity element with respect to \circ).

New target development: ONE-BT-1.

New development morphism: FUN-COMP-1-to-ONE-BT-1.

We can obtain the theorem that all transformation monoids are monoids almost for free by transporting results from MON-1 to ONE-BT-1. We start by creating the theory morphism from MON to ONE-BT that maps

$$(M, \cdot_{(M \times M) \rightarrow M}, \mathbf{e}_M)$$

to

$$(S \rightarrow S, \circ_{((S \rightarrow S) \times (S \rightarrow S)) \rightarrow (S \rightarrow S)}, \text{id}_{S \rightarrow S}) :$$

Theory Translation Definition 6.6 (MON to ONE-BT)

Name: MON-to-ONE-BT.

Source theory: MON.

Target theory: ONE-BT.

Base type mapping:

1. $M \mapsto S \rightarrow S$.

Constant mapping:

1. $\cdot_{(M \times M) \rightarrow M} \mapsto \circ_{((S \rightarrow S) \times (S \rightarrow S)) \rightarrow (S \rightarrow S)}$.
2. $\mathbf{e}_M \mapsto \text{id}_{S \rightarrow S}$.

The theory translation MON-to-ONE-BT is normal so that it has no obligations of the first kind by [20, Lemma 13.10]. It has two obligations of the second kind. These are valid in ONE-BT by part 4 of Lemma B.2 since $\circ_{((S \rightarrow S) \times (S \rightarrow S)) \rightarrow (S \rightarrow S)}$ and $\text{id}_{S \rightarrow S}$ are function abstractions which are defined by [20, Axiom A5.11]. It has two obligations of the third kind corresponding to the two axioms of MON. The two obligations are Thm18

and Thm19, which are obviously valid in ONE-BT. Therefore, MON-to-ONE-BT is a theory morphism from MON to ONE-BT by the Morphism Theorem [20, Theorem 13.13].

We can transport Def1, the definition of $\text{submonoid}_{\{M\} \rightarrow o}$, and Thm4, the theorem that says all submonoids are monoids, to ONE-BT-1 via MON-to-ONE-BT by a group transportation module:

Group Transportation 6.7 (Transport of Def1 & Thm2 to ONE-BT-1)

Name: submonoids-via-MON-to-ONE-BT.

Source development: MON-1.

Target development: ONE-BT-1.

Development morphism: MON-to-ONE-BT.

Definitions and theorems:

Def1: $\text{submonoid}_{\{M\} \rightarrow o} =$
 $\lambda s : \{M\} . s \neq \emptyset_{\{M\}} \wedge (\cdot \upharpoonright_{s \times s} \downarrow (s \times s) \rightarrow s) \wedge e \in s$ (submonoid).

Thm4: $\forall s : \{M\} . \text{submonoid } s \Rightarrow \text{MONOID}(s, \cdot \upharpoonright_{s \times s}, e)$
(submonoids are monoids).

Transported definitions and theorems:

Def6 (Def1-via-MON-to-ONE-BT): $\text{trans-monoid}_{\{S \rightarrow S\} \rightarrow o} =$
 $\lambda s : \{S \rightarrow S\} .$
 $s \neq \emptyset_{\{S \rightarrow S\}} \wedge$
 $(\circ_{((S \rightarrow S) \times (S \rightarrow S)) \rightarrow (S \rightarrow S)} \upharpoonright_{s \times s} \downarrow (s \times s) \rightarrow s) \wedge$
 $\text{id}_{S \rightarrow S} \in s$ (transformation monoid).

Thm20 (Thm4-via-MON-to-ONE-BT):
 $\forall s : \{S \rightarrow S\} .$
 $\text{trans-monoid } s \Rightarrow \text{MONOID}(s, \circ_{((S \rightarrow S) \times (S \rightarrow S)) \rightarrow (S \rightarrow S)} \upharpoonright_{s \times s}, \text{id}_{S \rightarrow S})$
(transformation monoids are monoids).

New target development: ONE-BT-2.

New development morphism: MON-1-to-ONE-BT-2.

trans-monoid is a predicate that is true when it is applied to a set of functions of $S \rightarrow S$ that forms a transformation monoid. Thm20 says that every transformation monoid — including the full transformation monoid — is a monoid.

7 Monoid Actions

A *(left) monoid action* is a structure $(m, s, \cdot, e, \text{act})$ where (m, \cdot, e) is a monoid and $\text{act} : (m \times s) \rightarrow s$ is a function such that

$$(1) \ x \text{ act } (y \text{ act } z) = (x \cdot y) \text{ act } z$$

for all $x, y \in M$ and $z \in s$ and

$$(2) \ e \text{ act } z = z$$

for all $z \in s$. We say in this case that the monoid (m, \cdot, e) *acts on* the set s *by* the function act .

Let \mathbf{Z}_o be the formula

$$\text{MON-ACTION}(\mathbf{M}_{\{\alpha\}}, \mathbf{S}_{\{\beta\}}, \mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}, \mathbf{E}_\alpha, \mathbf{G}_{(\alpha \times \beta) \rightarrow \beta}),$$

where MON-ACTION is the abbreviation introduced by the notational definition given in Table 3. \mathbf{Z}_o asserts that $(m, s, \cdot, e, \text{act})$ is a monoid action where m is a set denoting $\mathbf{M}_{\{\alpha\}}$, s is a set denoting $\mathbf{S}_{\{\beta\}}$, $\cdot : (m \times m) \rightarrow m$ is a function denoting $\mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}$, e is an element of m denoting \mathbf{E}_α , and $\text{act} : (m \times s) \rightarrow s$ is a function denoting $\mathbf{G}_{(\alpha \times \beta) \rightarrow \beta}$.

A theory of monoid actions is defined as an extension of the theory of monoids:

Theory Extension 7.1 (Monoid Actions)

Name: MON-ACT.

Extends MON.

New base types: S .

New constants: $\text{act}_{(M \times S) \rightarrow S}$.

New axioms:

$$3. \ \forall x, y : M, s : S. \ x \text{ act } (y \text{ act } s) = (x \cdot y) \text{ act } s \quad (\text{act is compatible with } \cdot).$$

$$4. \ \forall s : S. \ e \text{ act } s = s \quad (\text{act is compatible with } e).$$

We begin a development of MON-ACT by adding the definitions and theorems below:

Development Definition 7.2 (Monoid Actions 1)

Name: MON-ACT-1.

Bottom theory: MON-ACT.

Definitions and theorems:

Thm21: $\text{MON-ACTION}(U_{\{M\}}, U_{\{S\}}, \cdot_{(M \times M) \rightarrow M}, \mathbf{e}_M, \text{act}_{(M \times S) \rightarrow S})$
(models of MON-ACT define monoid actions).

Thm22: $\text{TOTAL}(\text{act}_{(M \times S) \rightarrow S})$ (act is total).

Def7: $\text{orbit}_{S \rightarrow \{S\}} = \lambda s : S . \{t : S \mid \exists x : M . x \text{ act } s = t\}$ (orbit).

Def8: $\text{stabilizer}_{S \rightarrow \{M\}} = \lambda s : S . \{x : M \mid x \text{ act } s = s\}$ (stabilizer).

Thm23: $\forall s : S . \text{submonoid}(\text{stabilizer } s)$ (stabilizers are submonoids).

Thm21 states that each model of MON-ACTION defines a monoid action. Thm22 says that $\text{act}_{(M \times S) \rightarrow S}$ is total. Def7 and Def8 introduce the concepts of an orbit and a stabilizer. And Thm23 states that a stabilizer of a monoid action $(m, s, \cdot, e, \text{act})$ is a submonoid of the monoid (m, \cdot, e) . The power of this machinery — monoid actions with orbits and stabilizers — is low with arbitrary monoids but very high with groups, i.e., monoids in which every element has an inverse.

Monoid actions are common in monoid theory. We will present two important examples of monoid actions. The first is the monoid action (m, m, \cdot, e, \cdot) such that the monoid (m, \cdot, e) acts on the set m of its elements by its function \cdot . We formalize this by creating the theory morphism from MON-ACT to MON that maps

$$(M, S, \cdot_{(M \times M) \rightarrow M}, \mathbf{e}_M, \text{act}_{(M \times S) \rightarrow S})$$

to

$$(M, M, \cdot_{(M \times M) \rightarrow M}, \mathbf{e}_M, \cdot_{(M \times M) \rightarrow M}) :$$

Theory Translation Definition 7.3 (MON-ACT to MON)

Name: MON-ACT-to-MON.

Source theory: MON-ACT.

Target theory: MON.

Base type mapping:

1. $M \mapsto M$.
2. $S \mapsto M$.

Constant mapping:

1. $\cdot_{(M \times M) \rightarrow M} \mapsto \cdot_{(M \times M) \rightarrow M}$.
2. $e_M \mapsto e_M$.
3. $\text{act}_{(M \times S) \rightarrow S} \mapsto \cdot_{(M \times M) \rightarrow M}$.

It is an easy exercise to verify that MON-ACT-to-MON is a theory morphism.

We can now transport Thm21 from MON-ACT to MON-3 via MON-ACT-to-MON to show that the action of a monoid (m, \cdot, e) on m by \cdot is a monoid action:

Theorem Transportation 7.4 (Transport of Thm21 to MON-3)

Name: monoid-action-via-MON-ACT-to-MON.

Source development: MON-ACT.

Target development: MON-3.

Development morphism: MON-ACT-to-MON.

Theorem:

Thm21: MON-ACTION($U_{\{M\}}, U_{\{S\}}, \cdot_{(M \times M) \rightarrow M}, e_M, \text{act}_{(M \times S) \rightarrow S}$)
(models of MON-ACT define monoid actions).

Transported theorem:

Thm24 (Thm21-via-MON-ACT-to-MON):
MON-ACTION($U_{\{M\}}, U_{\{M\}}, \cdot_{(M \times M) \rightarrow M}, e_M, \cdot_{(M \times M) \rightarrow M}$)
(first example is a monoid action).

New target development: MON-4.

The second example is a standard transformation monoid (f, \circ, id) on s acting on s by the function that applies a transformation to a member of s . (Note that all the functions in f are total by virtue of the transformation monoid being standard.) We formalize this example as a theory morphism from MON-ACT to ONE-BT extended with a set constant that denotes a standard transformation monoid. Here is the extension with a set constant $F_{\{S \rightarrow S\}}$ and two axioms:

Theory Extension 7.5 (One Base Type with a Set Constant)

Name: ONE-BT-with-SC.

Extends ONE-BT.

New base types:

New constants: $F_{\{S \rightarrow S\}}$.

New axioms:

1. trans-monoid F (F forms a transformation monoid).
2. $\forall f : F . \text{TOTAL}(f)$ (the members of F are total functions).

And here is the theory morphism from MON-ACT to ONE-BT-with-SC that maps

$$(M, S, \cdot_{(M \times M) \rightarrow M}, \mathbf{e}_M, \mathbf{act}_{(M \times S) \rightarrow S})$$

to

$$(F_{\{S \rightarrow S\}}, S, \circ_{((S \rightarrow S) \times (S \rightarrow S)) \rightarrow (S \rightarrow S)} \upharpoonright_{F \times F}, \mathbf{id}_{S \rightarrow S}, \bullet_{((S \rightarrow S) \times S) \rightarrow S} \upharpoonright_{F \times S}) :$$

Theory Translation Definition 7.6 (MON-ACT to ONE-BT-with-SC)

Name: MON-ACT-to-ONE-BT-with-SC.

Source theory: MON-ACT.

Target theory: ONE-BT-with-SC.

Base type mapping:

1. $M \mapsto F_{\{S \rightarrow S\}}$.
2. $S \mapsto S$.

Constant mapping:

1. $\cdot_{(M \times M) \rightarrow M} \mapsto \circ_{((S \rightarrow S) \times (S \rightarrow S)) \rightarrow (S \rightarrow S)} \uparrow \mathbb{F} \times \mathbb{F}$.
2. $e_M \mapsto \text{id}_{S \rightarrow S}$.
3. $\text{act}_{(M \times S) \rightarrow S} \mapsto \bullet_{((S \rightarrow S) \times S) \rightarrow S} \uparrow \mathbb{F} \times S$.

The parametric pseudoconstant $\bullet_{((S \rightarrow S) \times S) \rightarrow S} \uparrow \mathbb{F} \times S$ is defined in Table 2. It is a straightforward exercise to verify that MON-ACT-to-ONE-BT-with-SC is a theory morphism.

We can now transport Thm21 from MON-ACT to ONE-BT-with-S via MON-ACT-to-ONE-BT-with-SC to show that a total transformation monoid (f, \circ, id) on s acting on s by the function that applies a (total) transformation to a member of s is a monoid action:

Theorem Transportation 7.7 (Trans. of Thm21 to ONE-BT-with-SC)

Name: monoid-action-via-MON-ACT-to-ONE-BT-with-SC.

Source development: MON-ACT.

Target development: ONE-BT-with-SC.

Development morphism: MON-ACT-to-ONE-BT-with-SC.

Theorem:

Thm21: $\text{MON-ACTION}(U_{\{M\}}, U_{\{S\}}, \cdot_{(M \times M) \rightarrow M}, e_M, \text{act}_{(M \times S) \rightarrow S})$
(models of MON-ACT define monoid actions).

Transported theorem:

Thm25 (Thm21-via-MON-ACT-to-ONE-BT-with-SC):
 $\text{MON-ACTION}(\mathbb{F}_{\{S \rightarrow S\}},$
 $U_{\{S\}},$
 $\circ_{((S \rightarrow S) \times (S \rightarrow S)) \rightarrow (S \rightarrow S)} \uparrow \mathbb{F} \times \mathbb{F},$
 $\text{id}_{S \rightarrow S},$
 $\bullet_{((S \rightarrow S) \times S) \rightarrow S} \uparrow \mathbb{F} \times S)$
(second example is a monoid action).

New target development: ONE-BT-with-SC-1.

8 Monoid Homomorphisms

Roughly speaking, a *monoid homomorphism* is a structure-preserving mapping from one monoid to another.

Let \mathbf{W}_o be the formula

$$\text{MON-HOMOM}(\mathbf{M}_{\{\alpha\}}^1, \mathbf{M}_{\{\beta\}}^2, \mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}^1, \mathbf{E}_{\alpha}^1, \mathbf{F}_{(\beta \times \beta) \rightarrow \beta}^2, \mathbf{E}_{\beta}^2, \mathbf{H}_{\alpha \rightarrow \beta}),$$

where MON-HOMOM is the abbreviation introduced by the notational definition given in Table 3. \mathbf{W}_o asserts that

$$(m_1, m_2, \cdot_1, e_1, \cdot_2, e_2, h)$$

is a structure, where m_1 is a set denoting $\mathbf{M}_{\{\alpha\}}^1$, m_2 is a set denoting $\mathbf{M}_{\{\alpha\}}^2$, $\cdot_1 : (m_1 \times m_1) \rightarrow m_1$ is a function denoting $\mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}^1$, e_1 is an element of m_1 denoting \mathbf{E}_{α}^1 , $\cdot_2 : (m_2 \times m_2) \rightarrow m_2$ is a function denoting $\mathbf{F}_{(\alpha \times \alpha) \rightarrow \alpha}^2$, e_2 is an element of m_2 denoting \mathbf{E}_{α}^2 , and $h : m_1 \rightarrow m_2$ is a function denoting $\mathbf{H}_{\alpha \rightarrow \beta}$, such that h is a monoid homomorphism from a monoid (m_1, \cdot_1, e_1) to a monoid (m_2, \cdot_2, e_2) .

The notion of a monoid homomorphism is captured in the theory MON-HOM:

Theory Definition 8.1 (Monoid Homomorphisms)

Name: MON-HOM.

Base types: M_1, M_2 .

Constants: $\cdot_{(M_1 \times M_1) \rightarrow M_1}, e_{M_1}, \cdot_{(M_2 \times M_2) \rightarrow M_2}, e_{M_2}, h_{M_1 \rightarrow M_2}$.

Axioms:

1. $\forall x, y, z : M_1 . x \cdot (y \cdot z) = (x \cdot y) \cdot z$ ($\cdot_{(M_1 \times M_1) \rightarrow M_1}$ is associative).
2. $\forall x : M_1 . e \cdot x = x \cdot e = x$ (e_{M_1} is an identity element).
3. $\forall x, y, z : M_2 . x \cdot (y \cdot z) = (x \cdot y) \cdot z$ ($\cdot_{(M_2 \times M_2) \rightarrow M_2}$ is associative).
4. $\forall x : M_2 . e \cdot x = x \cdot e = x$ (e_{M_2} is an identity element).
5. $\forall x, y : M_1 . h(x \cdot y) = (h x) \cdot (h y)$ (first homomorphism property).
6. $h e_{M_1} = e_{M_2}$ (second homomorphism property).

$h_{M_1 \rightarrow M_2}$ denotes a monoid homomorphism from the monoid denoted by

$$(M_1, \cdot_{(M_1 \times M_1) \rightarrow M_1}, e_{M_1})$$

to the monoid denoted by

$$(M_2, \cdot_{(M_2 \times M_2) \rightarrow M_2}, e_{M_2}).$$

Here is a simple development of MON-HOM:

Development Definition 8.2 (Monoid Homomorphisms 1)

Name: MON-HOM-1.

Bottom theory: MON-HOM.

Definitions and theorems:

Thm26:

MON-HOM($U_{\{M_1\}}$,
 $U_{\{M_2\}}$,
 $\cdot_{(M_1 \times M_1) \rightarrow M_1}$,
 e_{M_1} ,
 $\cdot_{(M_2 \times M_2) \rightarrow M_2}$,
 e_{M_2} ,
 $h_{M_1 \rightarrow M_2}$)
(model of MON-HOM define monoid homomorphisms).

Thm27: TOTAL($h_{M_1 \rightarrow M_2}$) ($h_{M_1 \rightarrow M_2}$ is total).

There are embeddings (i.e., theory morphisms whose mappings are injective) from MON to the two copies of MON within MON-HOM defined by the following two theory translation definitions:

Theory Translation Definition 8.3 (First MON to MON-HOM)

Name: first-MON-to-MON-HOM.

Source theory: MON.

Target theory: MON-HOM.

Base type mapping:

1. $M \mapsto M_1$.

Constant mapping:

1. $\cdot_{(M \times M) \rightarrow M} \mapsto \cdot_{(M_1 \times M_1) \rightarrow M_1}$.
2. $e_M \mapsto e_{M_1}$.

Theory Translation Definition 8.4 (Second MON to MON-HOM)

Name: second-MON-to-MON-HOM.

Source theory: MON.

Target theory: MON-HOM.

Base type mapping:

1. $M \mapsto M_2$.

Constant mapping:

1. $\cdot_{(M \times M) \rightarrow M} \mapsto \cdot_{(M_2 \times M_2) \rightarrow M_2}$.
2. $e_M \mapsto e_{M_2}$.

An example of a monoid homomorphism from the monoid denoted by

$$(M, \cdot_{(M \times M) \rightarrow M}, e_M)$$

to the monoid denoted by

$$(\{M\}, \odot_{(\{M\} \times \{M\}) \rightarrow \{M\}}, E_{\{M\}})$$

is the function that maps a member x of the denotation of M to the singleton $\{x\}$. This monoid homomorphism is formalized by the following development morphism:

Development Translation Definition 8.5 (MON-HOM to MON)

Name: MON-HOM-to-MON-4.

Source development: MON-HOM.

Target development: MON-4.

Base type mapping:

1. $M_1 \mapsto M$.
2. $M_2 \mapsto \{M\}$.

Constant mapping:

1. $\cdot_{(M_1 \times M_1) \rightarrow M_1} \mapsto \cdot_{(M \times M) \rightarrow M}$.
2. $e_{M_1} \mapsto e_M$.
3. $\cdot_{(M_2 \times M_2) \rightarrow M_2} \mapsto \odot_{(\{M\} \times \{M\}) \rightarrow \{M\}}$.
4. $e_{M_2} \mapsto E_{\{M\}}$.
5. $h_{M_1 \rightarrow M_2} \mapsto \lambda x : M . \{x\}$.

It is a straightforward exercise to verify that HOM-MON-to-MON-4 is a theory morphism.

We can now transport Thm26 from MON-HOM to MON-4 via MON-HOM-to-MON-4 to show the example is a monoid homomorphism:

Theorem Transportation 8.6 (Transport of Thm26 to MON-4)

Name: monoid-action-via-MON-HOM-to-MON-4.

Source development: MON-HOM.

Target development: MON-4.

Development morphism: MON-HOM-to-MON-4.

Theorem:

Thm26:

MON-HOM($U_{\{M_1\}}$,
 $U_{\{M_2\}}$,
 $\cdot_{(M_1 \times M_1) \rightarrow M_1}$,
 e_{M_1} ,
 $\cdot_{(M_2 \times M_2) \rightarrow M_2}$,
 e_{M_2} ,
 $h_{M_1 \rightarrow M_2}$)

(models of MON-HOM define monoid homomorphisms).

Transported theorem:

Thm28 (Thm26-via-MON-HOM-to-MON-4)

MON-HOM($U_{\{M\}}$,
 $U_{\{\{M\}\}}$,
 $\cdot_{(M \times M) \rightarrow M}$,
 e_M ,
 $\cdot_{\{M\} \times \{M\} \rightarrow \{M\}}$,
 $E_{\{M\}}$,
 $\lambda x : M . \{x\}$) (example is a monoid homomorphism).

New target development: MON-5.

9 Monoids over Real Number Arithmetic

We need machinery concerning real number arithmetic to express some concepts about monoids. For instance, an iterated product operator for monoids involves integers. To formalize these kinds of concepts, we need a theory of monoids that includes real number arithmetic. We present in [20, Chapter 12] COF, a theory of complete ordered fields. COF is categorical in the standard sense (see [20]). That is, it has a single standard model up to isomorphism that defines the structure of real number arithmetic.

We define a theory of monoids over COF by extending COF with the language and axioms of MON:

Theory Extension 9.1 (Monoids over COF)

Name: MON-over-COF.

Extends COF.

New base types: M .

New constants: $\cdot_{(M \times M) \rightarrow M}$, e_M .

New axioms:

19. $\forall x, y, z : M . x \cdot (y \cdot z) = (x \cdot y) \cdot z$ (\cdot is associative).

20. $\forall x : M . e \cdot x = x \cdot e = x$ (e is an identity element).

We can now define an iterated product operator for monoids in a development of MON-over-COF-1:

Development Definition 9.2 (Monoids over COF 1)

Name: MON-over-COF-1.

Bottom theory: MON-over-COF.

Definitions and theorems:

Def9: $\text{prod}_{R \rightarrow R \rightarrow (R \rightarrow M) \rightarrow M} =$
 $\text{I } f : Z_{\{R\}} \rightarrow Z_{\{R\}} \rightarrow (Z_{\{R\}} \rightarrow M) \rightarrow M .$
 $\forall m, n : Z_{\{R\}}, g : Z_{\{R\}} \rightarrow M . f m n g \simeq$
 $(m > n \mapsto e \mid (f m (n - 1) g) \cdot (g n))$ (iterated product).

Thm29: $\forall m : Z_{\{R\}}, g : Z_{\{R\}} \rightarrow M . \left(\prod_{i=m}^m g i \right) \simeq g m$
(trivial product).

Thm30: $\forall m, k, n : Z_{\{R\}}, g : Z_{\{R\}} \rightarrow M .$
 $m < k < n \Rightarrow \left(\prod_{i=m}^k g i \right) \cdot \left(\prod_{i=k+1}^n g i \right) \simeq \prod_{i=m}^n g i$
(extended iterated product).

$Z_{\{R\}}$ is a quasytype that denotes the set of integers. Def9 defines the iterated product operator, and Thm29 and Thm30 are two theorems about the operator. Notice that we are utilizing the special notation

$$\left(\prod_{i=M_R}^{N_R} \mathbf{A}_M \right)$$

for the iterated product defined in Table 1.

We can similarly define extensions of MON over COF. For example, here is a theory of commutative monoids over COF and a development of it:

Theory Extension 9.3 (Commutative Monoids over COF)

Name: COM-MON-over-COF.

Extends MON-over-COF.

New base types:

New constants:

New axioms:

21. $\forall x, y : M . x \cdot y = y \cdot x$ (\cdot is commutative).

Development Definition 9.4 (Commutative Monoids over COF 1)

Name: COM-MON-over-COF-1.

Bottom theory: COM-MON-over-COF.

Definitions and theorems:

$$\text{Thm31: } \forall m, n : Z_{\{R\}}, g, h : Z_{\{R\}} \rightarrow M . \\ \left(\prod_{i=m}^n g i \right) \cdot \left(\prod_{i=m}^n h i \right) \simeq \prod_{i=m}^n (g i) \cdot (h i) \\ \text{(product of iterated products).}$$

Notice that this theorem holds only if \cdot is commutative.

For another example, here is a theory of commutative monoid actions over COF and a development of it:

Theory Extension 9.5 (Commutative Monoid Actions over COF)

Name: COM-MON-ACT-over-COF.

Extends COM-MON-over-COF.

New base types: S .

New constants: $\text{act}_{(M \times S) \rightarrow S}$.

New axioms:

22. $\forall x, y : M, s : S . x \text{ act } (y \text{ act } s) = (x \cdot y) \text{ act } s$
(act is compatible with \cdot).
23. $\forall s : S . e \text{ act } s = s$
(act is compatible with e).

Development Definition 9.6 (Com. Monoid Actions over COF 1)

Name: COM-MON-ACT-over-COF-1.

Bottom theory: COM-MON-ACT-over-COF.

Definitions and theorems:

$$\text{Thm32: } \forall x, y : M, s : S . x \text{ act } (y \text{ act } s) = y \text{ act } (x \text{ act } s) \\ \text{(act has commutative-like property).}$$

10 Monoid Theory Applied to Strings

In this section we will show how we can apply the machinery of our monoid theory formalization to a theory of strings over an abstract alphabet. A string over an alphabet A is a finite sequence of values from A . The finite sequence s can be represented as a partial function $s : \mathbb{N} \rightarrow A$ such that, for some $n \in \mathbb{N}$, $s(m)$ is defined iff $m < n$.

In [20, Chapter 10] we introduce compact notation for finite (and infinite) sequences represented in this manner. The notation requires a system of natural numbers as defined in [20]. The development COF-dev-2 of the theory COF presented in [20, Chapter 12] includes a system of natural numbers. Therefore, we can define a theory of strings as an extension of COF plus a base type A that represents an abstract alphabet:

Theory Extension 10.1 (Strings)

Name: STR.

Extends COF.

New base types: A .

New constants:

New axioms:

Since STR is an extension of COF, we can assume that STR-1 is a development of STR that contains the 7 definitions of COF-dev-2 named as COF-Def1, ..., COF-Def7 and the 22 theorems of COF-dev-2 named as COF-Thm1, ..., COF-Thm22. We can extend STR-1 as follows to include the basic definitions and theorems of strings:

Development Extension 10.2 (Strings 2)

Name: STR-2.

Extends STR-1.

New definitions and theorems:

Def10: $\text{str}_{\{R \rightarrow A\}} = [A]$ (string quasitype).

Def11: $\epsilon_{R \rightarrow A} = []_{R \rightarrow A}$ (empty string).

Def12: $\text{cat}_{((R \rightarrow A) \times (R \rightarrow A)) \rightarrow (R \rightarrow A)} =$
 $\text{I } f : (\text{str} \times \text{str}) \rightarrow \text{str} .$
 $\forall x : \text{str} . f(\epsilon, x) = x \wedge$
 $\forall a : A, x, y : \text{str} . f(a :: x, y) = a :: f(x, y). \quad (\text{concatenation}).$

Thm33: $\forall x : \text{str} . \epsilon x = x \epsilon = x \quad (\epsilon \text{ is an identity element}).$

Thm34: $\forall x, y, z : \text{str} . x(yz) = (xy)z \quad (\text{cat is associative}).$

Def10–Def12 utilize the compact notation introduced in Table 10.1 in [20] and Thm33–Thm34 utilize the compact notation introduced in Table 1.

We can define a development translation from MON-over-COF to STR-2 as follows:

Development Translation Definition 10.3 (MON-over-COF to STR-2)

Name: MON-over-COF-to-STR-2.

Source development: MON-over-COF.

Target development: STR-2.

Base type mapping:

1. $R \mapsto R.$
2. $M \mapsto \text{str}_{\{R \rightarrow A\}}.$

Constant mapping:

1. $0_R \mapsto 0_R.$
- \vdots
10. $\text{lub}_{R \rightarrow \{R\} \rightarrow o} \mapsto \text{lub}_{R \rightarrow \{R\} \rightarrow o}.$
11. $\cdot_{(M \times M) \rightarrow M} \mapsto \text{cat}_{((R \rightarrow A) \times (R \rightarrow A)) \rightarrow (R \rightarrow A)}.$
12. $e_M \mapsto \epsilon_{R \rightarrow A}.$

MON-over-COF-to-STR-2 has one obligation of the first kind for the mapped base type M , which is clearly valid since $\text{str}_{\{R \rightarrow A\}}$ is nonempty. MON-over-COF-to-STR-2 has 12 obligations of the second kind for the 12 mapped constants. The first 10 are trivially valid. The last 2 are valid by

Def12 and Def11, respectively. And MON-over-COF-to-STR-2 has 20 obligations of the third kind for the 20 axioms of MON-over-COF. The first 18 are trivially valid. The last 2 are valid by Thm34 and Thm33, respectively. Therefore, MON-over-COF-to-STR-2 is a development morphism from the theory MON-over-COF to the development STR-2 by the Morphism Theorem [20, Theorem 13.13].

The development morphism MON-over-COF-to-STR-2 allows us to transport definitions and theorems about monoids to the development STR-2. Here are five examples transported as a group:

Group Transportation 10.4 (Transport to STR-2)

Name: monoid-machinery-via-MON-over-COF-1-to-STR-2.

Source development: MON-over-COF-1.

Target development: STR-2.

Development morphism: MON-over-COF-to-STR-2.

Definitions and theorems:

Thm1: $\text{MONOID}(U_{\{M\}}, \cdot_{(M \times M) \rightarrow M}, e_M)$
(models of MON define monoids).

Def3: $\odot_{(\{M\} \times \{M\}) \rightarrow \{M\}} = \text{set-op}_{((M \times M) \rightarrow M) \rightarrow ((\{M\} \times \{M\}) \rightarrow \{M\})}$
(set product).

Def4: $E_{\{M\}} = \{e_M\}$ (set identity element).

Thm12 (Thm1-via-MON-to-set-monoid):
 $\text{MONOID}(U_{\{\{M\}\}}, \odot_{(\{M\} \times \{M\}) \rightarrow \{M\}}, E_{\{M\}})$
(set monoids are monoids).

Def9: $\text{prod}_{R \rightarrow R \rightarrow (R \rightarrow M) \rightarrow M} =$
 $I f : Z_{\{R\}} \rightarrow Z_{\{R\}} \rightarrow (Z_{\{R\}} \rightarrow M) \rightarrow M .$
 $\forall m, n : Z_{\{R\}}, g : Z_{\{R\}} \rightarrow M . f m n g \simeq$
 $(m > n \mapsto e \mid (f m (n - 1) g) \cdot (g n))$ (iterated product).

Transported definitions and theorems:

Thm35 (Thm1-via-MON-over-COF-to-STR-2):
 $\text{MONOID}(\text{str}_{\{R \rightarrow A\}}, \text{cat}_{((R \rightarrow A) \times (R \rightarrow A)) \rightarrow (R \rightarrow A)}, \epsilon_{R \rightarrow A})$
(strings form a monoid).

Def13 (Def3-via-MON-over-COF-to-STR-2):

$$\begin{aligned} \text{set-cat}_{(\{R \rightarrow A\} \times \{R \rightarrow A\}) \rightarrow \{R \rightarrow A\}} &= \\ \text{set-op}_{(((R \rightarrow A) \times (R \rightarrow A)) \rightarrow (R \rightarrow A)) \rightarrow ((\{R \rightarrow A\} \times \{R \rightarrow A\}) \rightarrow \{R \rightarrow A\})} &\text{cat} \\ &\text{(set concatenation)}. \end{aligned}$$

Def14 (Def4-via-MON-over-COF-to-STR-2):

$$E_{\{R \rightarrow A\}} = \{\epsilon_{R \rightarrow A}\} \quad \text{(set identity element).}$$

Thm36 (Thm12-via-MON-over-COF-1-to-STR-2):

$$\text{MONOID}(\mathcal{P}(\text{str}_{\{R \rightarrow A\}}), \text{set-cat}_{(\{R \rightarrow A\} \times \{R \rightarrow A\}) \rightarrow \{R \rightarrow A\}}, E_{\{R \rightarrow A\}}) \quad \text{(string sets form a monoid).}$$

Def15 (Def9-via-MON-over-COF-1-to-STR-2):

$$\begin{aligned} \text{iter-cat}_{R \rightarrow R \rightarrow (R \rightarrow (R \rightarrow A)) \rightarrow (R \rightarrow A)} &= \\ \text{I } f : Z_{\{R\}} \rightarrow Z_{\{R\}} \rightarrow (Z_{\{R\}} \rightarrow (R \rightarrow A)) \rightarrow (R \rightarrow A) &. \\ \forall m, n : Z_{\{R\}}, g : Z_{\{R\}} \rightarrow (R \rightarrow A) . f m n g \simeq & \\ (m > n \mapsto \epsilon \mid (f m (n - 1) g) \text{ cat } (g n)) & \\ &\text{(iterated concatenation)}. \end{aligned}$$

New target development: STR-3.

New development morphism: MON-over-COF-1-to-STR-3.

Notation for the application of

$$\text{set-cat}_{(\{R \rightarrow A\} \times \{R \rightarrow A\}) \rightarrow \{R \rightarrow A\}}$$

and

$$\text{iter-cat}_{R \rightarrow R \rightarrow (R \rightarrow (R \rightarrow A)) \rightarrow (R \rightarrow A)}$$

are defined in Table 1.

11 Related Work

As we have seen, a theory (or development) graph provides an effective architecture for formalizing a body of mathematical knowledge. It is especially useful for creating a large library of formal mathematical knowledge that, by necessity, must be constructed in parallel by multiple developers. The library is built in parts by separate development teams and then the parts are linked together by morphisms. Mathematical knowledge is organized as a theory graph in several proof assistants and logical frameworks including

Ergo [35], IMPS [21], Isabelle [5], LF [44], MMT [43], and PVS [38]. Theory graphs are also employed in several software specification and development systems including ASL [47], CASL [3, 4], EHDM [45], Hets [32], IOTA [33], KIDS [48], OBJ [24], and Specware [49].

Simple type theory in the form of Church’s type theory is a popular logic for formal mathematics. There are several proof assistants that implement versions of Church’s type theory including HOL [26], HOL Light [27], IMPS [22], Isabelle/HOL [39], ProofPower [42], PVS [37], and TPS [2]. As we mentioned in Section 1, the IMPS proof assistant is especially noteworthy here since it implements LUTINS [13, 14, 15], a version of Church’s type theory that admits undefined expressions and is closely related to Alonzo.

In recent years, there has been growing interest in formalizing mathematics within dependent logics. Several proof assistants and programming languages are based on versions of dependent type theory including Agda [7, 36], Automath [34], Coq [50], Epigram [11], F* [12], Idris [29], Lean [10], and Nuprl [9]. So which type theory is better for formal mathematics, simple type theory or dependent type theory? This question has become hotly contested. We hope that the reader will see our formalization of monoid theory in Alonzo as evidence for the efficacy of simple type theory as a logical basis for formal mathematics. The reader might also be interested in looking at these recent papers that advocate for simple type theory: [6, 40, 41].

Since monoid theory is a relatively simple subject, there have not been many attempts to formalize it by itself, but there have been several formalizations of group theory. Here are some examples: [23, 25, 30, 46, 51, 52].

12 Conclusion

The developments and development morphisms presented in Sections 2–10 form the development graph G_{mon} shown in Figure 1. The development graph shows all the development morphisms that we have explicitly defined as well as all inclusions corresponding to theory extensions we have explicitly defined. A development morphism that is an inclusion is designated by a \hookrightarrow arrow and a noninclusion is designated by a \rightarrow arrow. There are many, many more useful development morphisms that are not shown in G_{mon} , including implicit inclusions and a vast number of development morphisms into the theory COF.

The construction of G_{mon} illustrates how a body of mathematical knowledge can be formalized in Alonzo as a development graph in accordance with the little theories method. G_{mon} could be extended to include other

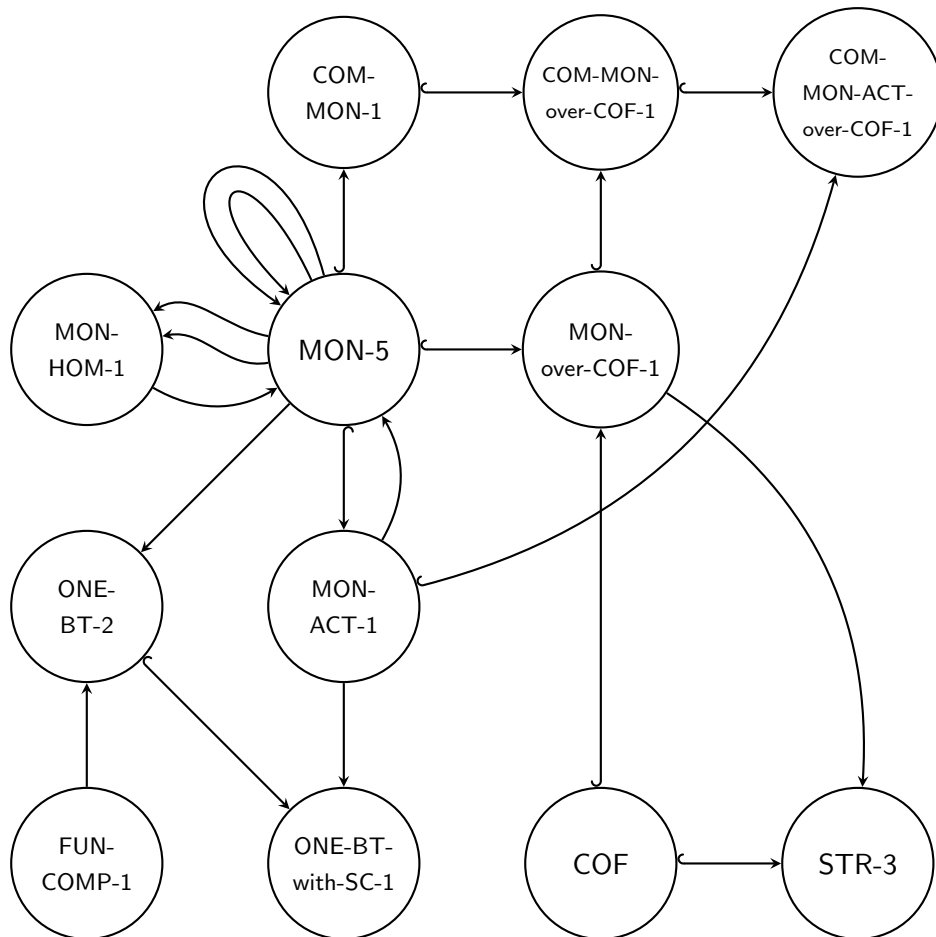


Figure 1: The Monoid Theory Development Graph

mathematical concepts related to monoids such as categories. It could be incorporated in a development graph that formalizes a more extensive body of mathematical knowledge. And it could also be used as a foundation for building a formalization of group theory. This would be done by lifting each development D of a theory T that extends MON to a development D' of a theory T' that extends a theory GRP obtained by adding an inverse operation to MON . The lifting of D to D' would include constructing inclusions from MON to GRP and from T to T' .

The formalization of monoid theory we have presented demonstrates three things. First, it demonstrates the power of the little theories method.

The formalization is largely free of redundancy since each mathematical topic is articulated in just one development D , the development for the little theory that is optimal for the topic in level of abstraction and choice of vocabulary. If we create a translation Φ from D to another development D' and prove that Φ is a morphism, then we can freely transport the definitions and theorems of D to D' via Φ . That is, an abstract concept or fact that has been validated in D can be translated to a concrete instance of the concept or fact that is automatically validated in D' provided the translation is a morphism. (This is illustrated by our use of the development morphism `MON-over-COF-to-STR-2` to transport definitions and theorems about monoids to a development about strings.) As the result, the same concept or fact can appear in many places in the theory graph but under different assumptions and involving different vocabulary. (For example, the notion of a submonoid represented by the constant `submonoid{M}→o` defined in `MON-1` appears in `ONE-BT-2` as the notion of a transformation monoid represented by the constant `trans-monoid{S→S}→o`.)

Second, the formalization demonstrates that Alonzo is well suited for expressing and reasoning about mathematical ideas. The simple type theory machinery of Alonzo — function and product types, function application and abstraction, definite description, and ordered pairs — enables mathematical expressions to be formulated in a direct and natural manner. It also enables almost every single mathematical structure or set of mathematical structures to be specified by an Alonzo development. (For example, the development `ONE-BT-2` specifies the set of structures consisting of a set S and the set $S \rightarrow S$ of transformations on S .) The admission of undefined expressions in Alonzo enables statements involving partial and total functions and definite descriptions to be expressed directly, naturally, and succinctly. (For example, if $M = (m, \cdot, e)$ is a monoid, the operation that makes a submonoid $m' \subseteq m$ of M a monoid itself is exactly what is expected: the partial function that results from restricting \cdot to $m' \times m'$.) And the notational definitions and conventions employed in Alonzo enables mathematical expressions to be presented with the same notation that is used mathematical practice. (For example, `Thm33`: $\forall x : \text{str} . \epsilon x = x \epsilon = x$, that states ϵ is an identity element for concatenation, is written just as one would expect it to be written in mathematical practice.)

Third, the formalization demonstrates that the communication-oriented approach to formal mathematics (with traditional proofs) has two advantages over the certification-oriented approach (with formal proofs): (1) communication is more effective and (2) formalization is easier. (These two approaches are defined in Section 1.) The certification-oriented approach

is done with the help of a proof assistant and all proofs are formal and mechanically checked. Proof assistants are consequently very complex and notoriously difficult to learn how to use. Traditional proofs are easier to read and write than formal proofs and are better suited for communicating the ideas behind proofs. Moreover, since the communication-oriented approach does not require a facility for developing and checking formal proofs, it can be done with software support that is much simpler and easier to use than a proof assistant. (In this paper, our software support was just a set of LaTeX macros and environments.)

The great majority of mathematics practitioners — including mathematicians — are much more interested in communicating mathematical ideas than in formally certifying mathematical results. Hence, the communication-oriented approach is likely to serve the needs of the average mathematics practitioner much better than the certification-oriented approach. This is especially true when the mathematical knowledge involved is well understood (such as monoid theory) and certification via traditional proof is adequate for the purpose at hand.

A Validation of Definitions and Theorems

Let $D = (T, \Xi)$ be a development where T is the bottom theory of the development and $\Xi = [P_1, \dots, P_n]$ is the list of definition and theorem packages of the development. For each i with $1 \leq i \leq n$, P_i has the form $(p, \mathbf{c}_\alpha, \mathbf{A}_\alpha, \pi)$ if P_i is a definition package and has the form $P_i = (p, \mathbf{A}_o, \pi)$ if P_i is a theorem package. Define $T_0 = T$ and, for all i with $0 \leq i \leq n - 1$, define $T_{i+1} = T[P_{i+1}]$ if P_{i+1} is a definition package and $T_{i+1} = T_i$ if P_{i+1} is a theorem package. In the former case, π_{i+1} is a proof that $\mathbf{A}_{\alpha\downarrow}$ is valid in T_i , and in the latter case, π_{i+1} is a proof that \mathbf{A}_o is valid in T_i . These proofs may be either traditional or formal.

The validation proofs for the definitions and theorems of a development are not included in the modules we have used to construct developments and to transport definitions and theorems. Instead, we give in this appendix, for each of the definitions and theorems in the developments defined in Sections 2–10, a traditional proof that validates the definition or theorem. The proofs are almost entirely straightforward. The proofs reference some of the axioms, rules of inference, and metatheorems of \mathfrak{A} , the formal proof system for Alonzo presented in [20]. These are legitimate to use since \mathfrak{A} is sound by the Soundness Theorem [20, Theorem 8.6 and Theorem B.11].

A.1 Development of MON

1. Thm1: $\text{MONOID}(U_{\{M\}}, \cdot_{(M \times M) \rightarrow M}, \mathbf{e}_M)$
 (models of MON define monoids).

Proof of the theorem. Let $T = (L, \Gamma)$ be MON. We must show

$$(\star) T \models \text{MONOID}(U_{\{M\}}, \cdot_{(M \times M) \rightarrow M}, \mathbf{e}_M).$$

$$\Gamma \models U_{\{M\}} \downarrow \tag{1}$$

$$\Gamma \models U_{\{M\}} \neq \emptyset_{\{M\}} \tag{2}$$

$$\Gamma \models \cdot_{(M \times M) \rightarrow M} \downarrow (U_{\{M\}} \times U_{\{M\}}) \rightarrow U_{\{M\}} \tag{3}$$

$$\Gamma \models \mathbf{e}_M \downarrow U_{\{M\}} \tag{4}$$

$$\Gamma \models \forall x, y, z : U_{\{M\}} . x \cdot (y \cdot z) = (x \cdot y) \cdot z \tag{5}$$

$$\Gamma \models \forall x : U_{\{M\}} . \mathbf{e} \cdot x = x \tag{6}$$

$$\Gamma \models \text{MONOID}(U_{\{M\}}, \cdot_{(M \times M) \rightarrow M}, \mathbf{e}_M) \tag{7}$$

(1) and (2) follow from parts 1 and 2, respectively, of Lemma B.1; (3) follows from [20, Axiom A5.2] and parts 8–10 of Lemma B.1; (4) follows from [20, Axiom A5.2] and part 8 of Lemma B.1; (5) and (6) follow from Axioms 1 and 2, respectively, of T and part 5 of Lemma B.1; and (7) follows from (1)–(6) and the definition of MONOID in Table 3. Therefore, (\star) holds. \square

2. Thm2: $\text{TOTAL}(\cdot_{(M \times M) \rightarrow M})$ (\cdot is total).

Proof of the theorem.

Let \mathbf{A}_o be

$$\forall x : M \times M . (\cdot_{(M \times M) \rightarrow M} x) \downarrow$$

and $T = (L, \Gamma)$ be MON. TOTAL is the abbreviation introduced by the notational definition given in Table 6.8 of [20], and so $\text{TOTAL}(\cdot_{(M \times M) \rightarrow M})$ stands for \mathbf{A}_o . Thus we must show $(\star) T \models \mathbf{A}_o$.

$$\Gamma \vDash (x : M \times M) \downarrow \quad (1)$$

$$\Gamma \vDash (x : M \times M) = (\text{fst } x, \text{snd } x) \quad (2)$$

$$\Gamma \vDash (\text{fst } x) \downarrow \wedge (\text{snd } x) \downarrow \quad (3)$$

$$\Gamma \vDash (\text{fst } x) \cdot ((\text{fst } x) \cdot (\text{snd } x)) = ((\text{fst } x) \cdot (\text{fst } x)) \cdot (\text{snd } x) \quad (4)$$

$$\Gamma \vDash ((\text{fst } x) \cdot (\text{snd } x)) \downarrow \quad (5)$$

$$\Gamma \vDash (\cdot_{(M \times M) \rightarrow M} ((\text{fst } x), (\text{snd } x))) \downarrow \quad (6)$$

$$\Gamma \vDash \mathbf{A}_o \quad (7)$$

(1) follows from variables always being defined by [20, Axiom A5.1]; (2) follows from (1) and [20, Axiom A7.4] by Universal Instantiation [20, Theorem A.14]; (3) follows from (2) by [20, Axioms A5.5, A7.2, and A7.3]; (4) follows from (3) and Axiom 1 of T by Universal Instantiation [20, Theorem A.14]; (5) follows from (4) by [20, Axioms A5.4 and A5.10]; (6) follows from (5) by notational definition; and (7) follows from (6) by Universal Generalization [20, Theorem A.30] using (2) and the fact that $(x : (M \times M))$ is not free in Γ since Γ is a set of sentences. \square

3. **Thm3:** $\forall x : M . (\forall y : M . x \cdot y = y \cdot x = y) \Rightarrow x = \mathbf{e}$
(uniqueness of identity element).

Proof of the theorem. Let \mathbf{A}_o be

$$\forall y : M . (x : M) \cdot y = y \cdot (x : M) = y$$

and $T = (L, \Gamma)$ be MON. We must show $(\star) T \vDash \forall x : M . \mathbf{A}_o \Rightarrow x = \mathbf{e}$.

$$\Gamma \cup \{\mathbf{A}_o\} \vDash \mathbf{e} \downarrow \quad (1)$$

$$\Gamma \cup \{\mathbf{A}_o\} \vDash (x : M) \downarrow \quad (2)$$

$$\Gamma \cup \{\mathbf{A}_o\} \vDash (x : M) \cdot \mathbf{e} = \mathbf{e} \cdot (x : M) = \mathbf{e} \quad (3)$$

$$\Gamma \cup \{\mathbf{A}_o\} \vDash \mathbf{e} \cdot (x : M) = (x : M) \cdot \mathbf{e} = (x : M) \quad (4)$$

$$\Gamma \cup \{\mathbf{A}_o\} \vDash (x : M) = \mathbf{e} \quad (5)$$

$$\Gamma \vDash \mathbf{A}_o \Rightarrow (x : M) = \mathbf{e} \quad (6)$$

$$\Gamma \vDash \forall x : M . \mathbf{A}_o \Rightarrow x = \mathbf{e} \quad (7)$$

(1) follows from constants always being defined by [20, Axiom A5.2]; (2) follows from variables always being defined by [20, Axiom A5.1]; (3) follows (1) and \mathbf{A}_o by Universal Instantiation [20, Theorem A.14];

(4) follows (2) and Axiom 2 of T by Universal Instantiation; (5) follows from (3) and (4) by the Equality Rules [20, Lemma A.13]; (6) follows from (5) by the Deduction Theorem [20, Lemma A.50]; and (7) follows from (6) by Universal Generalization [20, Theorem A.30] using the fact that $(x : M)$ is not free in Γ since Γ is a set of sentences. Therefore, (\star) holds. \square

4. Def1: $\text{submonoid}_{\{M\} \rightarrow o} =$
 $\lambda s : \{M\} . s \neq \emptyset_{\{M\}} \wedge (\cdot \upharpoonright_{s \times s} \downarrow (s \times s) \rightarrow s) \wedge \mathbf{e} \in s$ (submonoid).

Proof that RHS is defined. Let $\mathbf{A}_{\{M\} \rightarrow o}$ be the RHS of Def1. We must show that $\text{MON} \models \mathbf{A}_{\{M\} \rightarrow o} \downarrow$. This follows immediately from function abstractions always being defined by [20, Axiom A5.11]. \square

5. Thm4: $\forall s : \{M\} . \text{submonoid } s \Rightarrow \text{MONOID}(s, \cdot \upharpoonright_{s \times s}, \mathbf{e})$
(submonoids are monoids).

Proof of the theorem. Let \mathbf{A}_o be

$$\text{submonoid}(s)$$

and $T = (L, \Gamma)$ be MON extended by Def1. We must show

$$(\star) T \models \forall s : \{M\} . \mathbf{A}_o \Rightarrow \text{MONOID}(s, \cdot \upharpoonright_{(s \times s)}, \mathbf{e}).$$

$$\Gamma \cup \{\mathbf{A}_o\} \models s_{\{M\}} \downarrow \tag{1}$$

$$\Gamma \cup \{\mathbf{A}_o\} \models s \neq \emptyset_{\{M\}} \tag{2}$$

$$\Gamma \cup \{\mathbf{A}_o\} \models \cdot \upharpoonright_{(s \times s)} \downarrow (s \times s) \rightarrow s \tag{3}$$

$$\Gamma \cup \{\mathbf{A}_o\} \models \mathbf{e} \in s \tag{4}$$

$$\Gamma \cup \{\mathbf{A}_o\} \models \mathbf{e} \downarrow s \tag{5}$$

$$\begin{aligned} \Gamma \cup \{\mathbf{A}_o\} \models \forall x, y, z : s . \cdot \upharpoonright_{(s \times s)} (x, \cdot \upharpoonright_{(s \times s)} (y, z)) \\ = \cdot \upharpoonright_{(s \times s)} (\cdot \upharpoonright_{(s \times s)} (x, y), z) \end{aligned} \tag{6}$$

$$\Gamma \cup \{\mathbf{A}_o\} \models \forall x : s . \cdot \upharpoonright_{(s \times s)} (\mathbf{e}, x) = \cdot \upharpoonright_{(s \times s)} (x, \mathbf{e}) = x \tag{7}$$

$$\Gamma \cup \{\mathbf{A}_o\} \models \text{MONOID}(s, \cdot \upharpoonright_{(s \times s)}, \mathbf{e}) \tag{8}$$

$$\Gamma \models \mathbf{A}_o \Rightarrow \text{MONOID}(s, \cdot \upharpoonright_{(s \times s)}, \mathbf{e}) \tag{9}$$

$$\Gamma \models \forall s : \{M\} . \mathbf{A}_o \Rightarrow \text{MONOID}(s, \cdot \upharpoonright_{(s \times s)}, \mathbf{e}) \tag{10}$$

(1) follows from variables always being defined by [20, Axiom A5.1]; (2), (3), and (4) follow directly from Def1; (5) follows from [20, Axiom

A5.2] and (4); (6) and (7) follow from **Thm1**, $\cdot \upharpoonright_{(s \times s)} \sqsubseteq \cdot_{(M \times M) \rightarrow M}$, and the fact that $\cdot \upharpoonright_{(s \times s)}$ is total by **Thm2**; (8) follows from (1)–(3) and (5)–(7) by the definition of **MONOID** in Table 3; (9) follows from (8) by the Deduction Theorem [20, Theorem A.50]; and (10) follows from (9) by Universal Generalization [20, Theorem A.30] using the fact that $(s : \{M\})$ is not free in Γ since Γ is a set of sentences. Therefore, (\star) holds. \square

6. **Thm5: submonoid $\{e\}$** (minimum submonoid).

Proof of the theorem. Let $T = (L, \Gamma)$ be **MON** extended by **Def1**. We must show $(\star) T \models \text{submonoid } \{e\}$.

$$\Gamma \models e \in \{e\} \quad (1)$$

$$\Gamma \models \{e\} \neq \emptyset_{\{M\}} \quad (2)$$

$$\Gamma \models e \cdot e = e \quad (3)$$

$$\Gamma \models \cdot \upharpoonright_{\{e\} \times \{e\}} \downarrow (\{e\} \times \{e\}) \rightarrow \{e\} \quad (4)$$

$$\Gamma \models \text{submonoid } \{e\} \quad (5)$$

(1) is trivial; (2) follows from (1) because $\{e\}$ has at least one member; (3) follows from Axiom 2 of T by Universal Instantiation [20, Theorem A.14]; (4) follows directly from (1), (3), and the fact that the only member of $\{e\}$ is e ; and (5) follows from (1), (2), (4), and **Def1**. Therefore, (\star) holds. \square

7. **Thm6: submonoid $U_{\{M\}}$** (maximum submonoid).

Proof of the theorem. Let $T = (L, \Gamma)$ be **MON** extended by **Def1**. We must show $(\star) T \models \text{submonoid } U_{\{M\}}$.

$$\Gamma \models \text{MONOID}(U_{\{M\}}, \cdot_{(M \times M) \rightarrow M}, e) \quad (1)$$

$$\Gamma \models U_{\{M\}} \neq \emptyset_{\{M\}} \wedge e \in U_{\{M\}} \quad (2)$$

$$\Gamma \models \cdot \upharpoonright_{U_{\{M\}} \times U_{\{M\}}} \downarrow (U_{\{M\}} \times U_{\{M\}}) \rightarrow U_{\{M\}} \quad (3)$$

$$\Gamma \models \text{submonoid } U_{\{M\}} \quad (4)$$

(1) is **Thm1**; (2) follows immediately from (1); (3) follows from (1) by part 12 of Lemma **B.1**; and (4) follows from (1), (2), (3), and **Def1**. . Therefore, (\star) holds. \square

8. Def2: $\cdot^{\text{op}}_{(M \times M) \rightarrow M} = \lambda p : M \times M . (\text{snd } p) \cdot (\text{fst } p)$ (opposite of \cdot).

Proof that RHS is defined. Similar to the proof that the RHS of Def1 is defined. \square

9. Thm7: $\forall x, y, z : M . x \cdot^{\text{op}} (y \cdot^{\text{op}} z) = (x \cdot^{\text{op}} y) \cdot^{\text{op}} z$ (\cdot^{op} is associative).

Proof of the theorem. Let \mathbf{A}_o be

$$x \cdot^{\text{op}} (y \cdot^{\text{op}} z) = (x \cdot^{\text{op}} y) \cdot^{\text{op}} z$$

and $T = (L, \Gamma)$ be MON extended by Def2. We must show

$$(\star) T \models \forall x, y, z : M . \mathbf{A}_o.$$

$$\Gamma \models (x : M) \downarrow \wedge (y : M) \downarrow \wedge (z : M) \downarrow \tag{1}$$

$$\Gamma \models (z \cdot y) \cdot x = z \cdot (y \cdot x) \tag{2}$$

$$\Gamma \models \mathbf{A}_o \tag{3}$$

$$\Gamma \models \forall x, y, z : M . \mathbf{A}_o \tag{4}$$

(1) follows from variables always being defined by [20, Axiom A5.1]; (2) follows from (1) and Axiom 1 of T by Universal Instantiation [20, Theorem A.14] and the Equality Rules [20, Theorem A.13]; (3) follows from Lemma B.4 and (2) by repeated applications of Rule R2' [20, Lemma A.2] using $(\star\star)$ the fact that $(x : M)$, $(y : M)$, and $(z : M)$ are not free in Γ since Γ is a set of sentences; and (4) follows from (3) by Universal Generalization [20, Theorem A.30] again using $(\star\star)$. Therefore (\star) holds. \square

10. Thm8: $\forall x : M . e \cdot^{\text{op}} x = x \cdot^{\text{op}} e = x$ (e is an identity element with respect to \cdot^{op}).

Proof of the theorem. Let \mathbf{A}_o be

$$e \cdot^{\text{op}} x = x \cdot^{\text{op}} e = x$$

and $T = (L, \Gamma)$ be MON extended by Def2. We must show

$$(\star) T \models \forall x : M . \mathbf{A}_o.$$

$$\Gamma \models (x : M) \downarrow \tag{1}$$

$$\Gamma \models x \cdot e = e \cdot x = x \tag{2}$$

$$\Gamma \models \mathbf{A}_o \tag{3}$$

$$\Gamma \models \forall x : M . \mathbf{A}_o \tag{4}$$

(1) follows from variables always being defined by [20, Axiom A5.1]; (2) follows from (1) and Axiom 2 of T by Universal Instantiation [20, Theorem A.14] and the Equality Rules [20, Theorem A.13]; (3) follows from Lemma B.4 and (2) by repeated applications of Rule R2' [20, Lemma A.2] using $(\star\star)$ the fact that $(x : M)$ is not free in Γ since Γ is a set of sentences; and (4) follows from (3) by Universal Generalization [20, Theorem A.30] again using $(\star\star)$. Therefore (\star) holds. \square

11. Def3: $\odot_{(\{M\} \times \{M\}) \rightarrow \{M\}} = \text{set-op}_{((M \times M) \rightarrow M) \rightarrow ((\{M\} \times \{M\}) \rightarrow \{M\})}$.
(set product).

Proof that RHS is defined. Similar to the proof that the RHS of Def1 is defined. \square

12. Def4: $E_{\{M\}} = \{e_M\}$ (set identity element).

Proof that RHS is defined. We must show that $\text{MON} \models \{e_M\} \downarrow$. Now $\{e_M\}$ stands for

$$(\lambda x_1 : M . \lambda x : M . x = x_1)(e_M).$$

Since constants are always defined by [20, Axiom A5.2], $\{e_M\}$ beta-reduces [20, Theorem 7.1 and Axiom A4] to

$$\lambda x : M . x = e_M.$$

Since every function abstraction is defined by [20, Axiom A5.11], we have $\{e_M\} \downarrow$ by Quasi-Equality Substitution [20, Lemma A.2]. \square

13. Thm9: $\forall x, y, z : \{M\} . x \odot (y \odot z) = (x \odot y) \odot z$ (\odot is associative).

Proof of the theorem. Let $T = (L, \Gamma)$ be MON extended by Def3. We must show

$$(\star) T \models \forall x, y, z : \{M\} . x \odot (y \odot z) = (x \odot y) \odot z.$$

$$\Gamma \models (x : \{M\}) \downarrow \wedge (y : \{M\}) \downarrow \wedge (z : \{M\}) \downarrow \quad (1)$$

$$\Gamma \models x \odot (y \odot z) = \{d : M \mid \exists a : x, b : y, c : z . d = a \cdot (b \cdot c)\} \quad (2)$$

$$\Gamma \models (x \odot y) \odot z = \{d : M \mid \exists a : x, b : y, c : z . d = (a \cdot b) \cdot c\} \quad (3)$$

$$\Gamma \models x \odot (y \odot z) = (x \odot y) \odot z \quad (4)$$

$$\Gamma \models \forall x, y, z : \{M\} . x \odot (y \odot z) = (x \odot y) \odot z \quad (5)$$

(1) follows from variables always being defined by [20, Axiom A5.1]; (2) and (3) follow from (1) and Def3; (4) follows from (2) and (3) by Axiom 1 of T ; and (5) follows from (4) by Universal Generalization [20, Theorem A.30] using the fact that x , y , and z are not free in Γ since Γ is a set of sentences. Therefore, (\star) holds. \square

14. Thm10: $\forall x : \{M\} . \mathbf{E} \odot x = x \odot \mathbf{E} = x$
 (\mathbf{E} is an identity element with respect to \odot).

Proof of the theorem. Let $T = (L, \Gamma)$ be MON extended by Def3 and Def4. We must show

$$(\star) T \models \forall x : \{M\} . \mathbf{E} \odot x = x \odot \mathbf{E} = x.$$

$$\Gamma \models (x : \{M\}) \downarrow \tag{1}$$

$$\Gamma \models \mathbf{E} \downarrow \tag{2}$$

$$\Gamma \models \mathbf{E} \odot x = \{b : M \mid \exists a : x . b = \mathbf{e} \cdot a\} \tag{3}$$

$$\Gamma \models x \odot \mathbf{E} = \{b : M \mid \exists a : x . b = a \cdot \mathbf{e}\} \tag{4}$$

$$\Gamma \models \mathbf{E} \odot x = x \odot \mathbf{E} = x \tag{5}$$

$$\Gamma \models \forall x : \{M\} . \mathbf{E} \odot x = x \odot \mathbf{E} = x \tag{6}$$

(1) follows from variables always being defined by [20, Axiom A5.1]; (2) follows from constants always defined by [20, Axiom A5.2]; (3) and (4) follow from (1), (2), Def3, and Def4; (5) follows from (3) and (4) by Axiom 2 of T ; and (6) follows from (5) by Universal Generalization [20, Theorem A.30] using the fact that x is not free in Γ since Γ is a set of sentences. Therefore, (\star) holds. \square

15. Thm11 (Thm1-via-MON-to-opposite-monoid):
 MONOID($U_{\{M\}}, \overset{\text{op}}{(M \times M) \rightarrow M}, \mathbf{e}_M$) (opposite monoids are monoids).

Proof of the theorem. Let T be the top theory of MON-1. We must show $T \models \text{Thm11}$. We have previously proved $(\star) \text{MON} \models \text{Thm1}$. $\Phi = \text{MON-to-opposite-monoid}$ is a development morphism from MON to MON-1, and so $\tilde{\Phi} = (\mu, \nu)$ is a theory morphism from MON to T . Thus (\star) implies $T \models \nu(\text{Thm1})$. Therefore, $T \models \text{Thm11}$ since $\text{Thm11} = \nu(\text{Thm1})$. \square

16. Thm12 (Thm1-via-MON-to-set-monoid):
 MONOID($U_{\{\{M\}\}}, \odot_{(\{M\} \times \{M\}) \rightarrow \{M\}}, \mathbf{E}_{\{M\}}$)
 (set monoids are monoids).

Proof of the theorem. Similar to the proof of Thm11. \square

17. Thm24 (Thm21-via-MON-ACT-to-MON):

MON-ACTION($U_{\{M\}}, U_{\{M\}}, \cdot_{(M \times M) \rightarrow M}, \mathbf{e}_M, \cdot_{(M \times M) \rightarrow M}$)
(first example is a monoid action).

Proof of the theorem. Similar to the proof of Thm11. \square

A.2 Development of COM-MON

1. Thm13: COM-MONOID($U_{\{M\}}, \cdot_{(M \times M) \rightarrow M}, \mathbf{e}_M$)
(models of COM-MON define commutative monoids).

Proof of the theorem. Let $T = (L, \Gamma)$ be COM-MON. We must show

(\star) $T \models \text{COM-MONOID}(U_{\{M\}}, \cdot_{(M \times M) \rightarrow M}, \mathbf{e}_M)$.

$\Gamma \models \text{MONOID}(U_{\{M\}}, \cdot_{(M \times M) \rightarrow M}, \mathbf{e}_M)$ (1)

$\Gamma \models \forall x, y : U_{\{M\}} . x \cdot y = y \cdot x$ (2)

(1) follows from $\text{MON} \leq T$ and the fact that Thm1 is a theorem of MON; and (2) follows from Axiom 3 of T and part 5 of Lemma B.1. Therefore, (\star) follows from (1), (2), and the notational definition of COM-MONOID given in Table 3. \square

2. Def5: $\leq_{M \rightarrow M \rightarrow o} = \lambda x, y : M . \exists z : M . x \cdot z = y$ (weak order).

Proof that RHS is defined. Similar to the proof that the RHS of Def1 is defined. \square

3. Thm14: $\forall x : M . x \leq x$ (reflexivity).

Proof of the theorem. Let $T = (L, \Gamma)$ be COM-MON extended by Def5. We must show

(\star) $T \models \forall x : M . x \leq x$.

$\Gamma \models (x : M) \downarrow$ (1)

$\Gamma \models (x \leq x) \simeq (\exists z : M . x \cdot z = x)$ (2)

$\Gamma \models x \cdot \mathbf{e} = x$ (3)

$\Gamma \models \exists z : M . x \cdot z = x$ (4)

$\Gamma \models x \leq x$ (5)

$\Gamma \models \forall x : M . x \leq x$ (6)

(1) follows from variables always being defined by [20, Axiom A5.1]; (2) follows from Def5 and Extensionality [20, Axiom A3] using the Substitution Rule [20, Theorem A.31] and Beta-Reduction [20, Theorem 7.1 and Axiom A4]; (3) follows from (1) and Axiom 2 of T by Universal Instantiation [20, Theorem A.14]; (4) follows from (3) by Existential Generalization [20, Theorem A.51]; (5) follows from (2) and (4) by Rule R2' [20, Lemma A.2]; and (6) follows from (5) by Universal Generalization [20, Theorem A.30] using the fact that x is not free in Γ since Γ is a set of sentences. Therefore, (\star) holds. \square

4. Thm15: $\forall x, y, z : M . (x \leq y \wedge y \leq z) \Rightarrow x \leq z$ (transitivity).

Proof of the theorem. Let \mathbf{A}_o be $(x \leq y \wedge y \leq z)$, \mathbf{B}_o be $x \cdot u = y$, and \mathbf{C}_o be $y \cdot v = z$ (where these variables all have type M). Also let $T = (L, \Gamma)$ be COM-MON extended by Def5. We must show

$$(\star) T \models \forall x, y, z : M . \mathbf{A}_o \Rightarrow x \leq z.$$

$$\Gamma \cup \{\mathbf{B}_o, \mathbf{C}_o\} \models (x : M) \downarrow \wedge (y : M) \downarrow \wedge (z : M) \downarrow \wedge (u : M) \downarrow \wedge (v : M) \downarrow \quad (1)$$

$$\Gamma \cup \{\mathbf{B}_o, \mathbf{C}_o\} \models (x \cdot u) \cdot v = z \quad (2)$$

$$\Gamma \cup \{\mathbf{B}_o, \mathbf{C}_o\} \models (x \cdot u) \cdot v = x \cdot (u \cdot v) \quad (3)$$

$$\Gamma \cup \{\mathbf{B}_o, \mathbf{C}_o\} \models x \cdot (u \cdot v) = z \quad (4)$$

$$\Gamma \cup \{\mathbf{B}_o, \mathbf{C}_o\} \models \exists w : M . x \cdot w = z \quad (5)$$

$$\Gamma \cup \{\mathbf{B}_o\} \models (y \cdot v = z) \Rightarrow (\exists w : M . x \cdot w = z) \quad (6)$$

$$\Gamma \cup \{\mathbf{B}_o\} \models (\exists v : M . y \cdot v = z) \Rightarrow (\exists w : M . x \cdot w = z) \quad (7)$$

$$\begin{aligned} \Gamma \models (x \cdot u = y) &\Rightarrow \\ &((\exists v : M . y \cdot v = z) \Rightarrow (\exists w : M . x \cdot w = z)) \end{aligned} \quad (8)$$

$$\begin{aligned} \Gamma \models (\exists u : M . x \cdot u = y) &\Rightarrow \\ &((\exists v : M . y \cdot v = z) \Rightarrow (\exists w : M . x \cdot w = z)) \end{aligned} \quad (9)$$

$$\Gamma \models x \leq y \Rightarrow (y \leq z \Rightarrow x \leq z) \quad (10)$$

$$\Gamma \models \mathbf{A}_o \Rightarrow x \leq z \quad (11)$$

$$\Gamma \models \forall x, y, z : M . \mathbf{A}_o \Rightarrow x \leq z \quad (12)$$

(1) follows from variables always being defined by [20, Axiom A5.1]; (2) follows from \mathbf{B}_o and \mathbf{C}_o by the Equality Rules [20, Lemma A.13];

(3) follows from Axiom 1 of T ; (4) follows from (2) and (3) by the Equality Rules [20, Lemma A.13]; (5) follows from (1), (4), and Thm2 by Existential Generalization [20, Theorem A.51]; (6) and (8) follow from (5) and (7), respectively, by the Deduction Theorem [20, Theorem A.50]; (7) and (9) follow from (6) and (8), respectively, by Existential Instantiation (Theorem B.3); (10) follows from (1), (9), and Def5 by Beta-Reduction [20, Theorem 7.1 and Axiom A4] and Alpha-Conversion [20, Theorem 7.6 and Theorem A.18]; (11) follows from (10) by the Tautology Rule [20, Corollary A.46]; and (12) follows from (11) by Universal Generalization [20, Theorem A.30] using the fact that x , y , and z are not free in Γ since Γ is a set of sentences. Therefore, (\star) holds. \square

A.3 Development of FUN-COMP

1. Thm16: $\forall f : A \rightarrow B, g : B \rightarrow C, h : C \rightarrow D . f \circ (g \circ h) = (f \circ g) \circ h$
(\circ is associative).

Proof of the theorem. Let \mathbf{A}_o be the theorem and $T = (L, \Gamma)$ be FUN-COMP. We must show $(\star) T \models \mathbf{A}_o$.

$$\Gamma \models (f : A \rightarrow B) \downarrow \wedge (g : B \rightarrow C) \downarrow \wedge (h : C \rightarrow D) \downarrow \wedge (x : A) \downarrow \quad (1)$$

$$\Gamma \models ((f \circ g) \circ h) x \simeq h(g(f x)) \quad (2)$$

$$\Gamma \models (f \circ (g \circ h)) x \simeq h(g(f x)) \quad (3)$$

$$\Gamma \models ((f \circ g) \circ h) x \simeq (f \circ (g \circ h)) x \quad (4)$$

$$\Gamma \models \forall x : A . (f \circ (g \circ h)) x \simeq ((f \circ g) \circ h) x \quad (5)$$

$$\Gamma \models f \circ (g \circ h) = (f \circ g) \circ h \quad (6)$$

$$\Gamma \models \mathbf{A}_o \quad (7)$$

(1) follows from variables always being defined by [20, Axiom A5.1]; (2) and (3) both follow from (1), the definition of \circ in Table 2, function abstractions are always defined by [20, Axiom A5.11], ordered pairs of defined components are always defined by [20, Axiom A7.1], Beta-Reduction [20, Theorem 7.1 and Axiom A4], and Quasi-Equality Substitution [20, Lemma A.2]; (4) follows from (2) and (3) by the Quasi-Equality Rules [20, Lemma A.4]; (5) follows from (4) by Universal Generalization [20, Theorem A.30] using the fact that x is not free in Γ since Γ is a set of sentences; (6) follows from (5) by Extensionality [20, Axiom A3]; and (7) follows from (6) by Universal Generalization using the fact that f , g , and h are not free in Γ since Γ is a set of sentences. Therefore, (\star) holds. \square

2. Thm17: $\forall f : A \rightarrow B . \text{id}_{A \rightarrow A} \circ f = f \circ \text{id}_{B \rightarrow B} = f$
(identity functions are left and right identity elements).

Proof of the theorem. Let \mathbf{A}_o be the theorem and $T = (L, \Gamma)$ be FUN-COMP. We must show $(\star) T \vDash \mathbf{A}_o$.

$$\Gamma \vDash (f : A \rightarrow B) \downarrow \wedge (x : A) \downarrow \quad (1)$$

$$\Gamma \vDash (\text{id}_{A \rightarrow A} \circ f) x \simeq f x \quad (2)$$

$$\Gamma \vDash (f \circ \text{id}_{B \rightarrow B}) x \simeq f x \quad (3)$$

$$\Gamma \vDash \forall x : A . (\text{id}_{A \rightarrow A} \circ f) x \simeq f x \quad (4)$$

$$\Gamma \vDash \forall x : A . (f \circ \text{id}_{B \rightarrow B}) x \simeq f x \quad (5)$$

$$\Gamma \vDash \text{id}_{A \rightarrow A} \circ f = f \quad (6)$$

$$\Gamma \vDash f \circ \text{id}_{B \rightarrow B} = f \quad (7)$$

$$\Gamma \vDash \text{id}_{A \rightarrow A} \circ f = f \circ \text{id}_{B \rightarrow B} = f \quad (8)$$

$$\Gamma \vDash \mathbf{A}_o \quad (9)$$

(1) follows from variables always being defined by [20, Axiom A5.1]; (2) and (3) both follow from (1), the definitions of id and \circ in Table 2, function abstractions are always defined by [20, Axiom A5.11], ordered pairs of defined components are always defined by [20, Axiom A7.1], Beta-Reduction [20, Theorem 7.1 and Axiom A4], and Quasi-Equality Substitution [20, Lemma A.2]; (4) and (5) both follow from (2) and (3), respectively, by Universal Generalization [20, Theorem A.30] using the fact that x is not free in Γ since Γ is a set of sentences; (6) and (7) follow from (4) and (5), respectively, by Extensionality [20, Axiom A3]; (8) follows from (6) and (7) by the Equality Rules [20, Lemma A.13]; and (9) follows from (8) by Universal Generalization using the fact that f is not free in Γ since Γ is a set of sentences. Therefore, (\star) holds. \square

A.4 Development of ONE-BT

1. Thm18 (Thm16-via-FUN-COMP-to-ONE-BT):
 $\forall f, g, h : S \rightarrow S . f \circ (g \circ h) = (f \circ g) \circ h$ (\circ is associative).

Proof of the theorem. Similar to the proof of Thm11. \square

2. Thm19 (Thm17-via-FUN-COMP-to-ONE-BT):
 $\forall f : S \rightarrow S . \text{id}_{S \rightarrow S} \circ f = f \circ \text{id}_{S \rightarrow S} = f$
($\text{id}_{S \rightarrow S}$ is an identity element with respect to \circ).

Proof of the theorem. Similar to the proof of Thm11. \square

3. Def6 (Def1-via-MON-to-ONE-BT):

$$\begin{aligned} & \text{trans-monoid}_{\{S \rightarrow S\} \rightarrow o} = \\ & \lambda s : \{S \rightarrow S\} . \\ & \quad s \neq \emptyset_{\{S \rightarrow S\}} \wedge \\ & \quad \text{TOTAL-ON}(\circ_{((S \rightarrow S) \times (S \rightarrow S)) \rightarrow (S \rightarrow S)} \downarrow_{s \times s}, s \times s, s) \wedge \\ & \quad \text{id}_{S \rightarrow S} \in s \quad (\text{transformation monoid}). \end{aligned}$$

Proof that RHS is defined. Let $\mathbf{A}_{\{M\} \rightarrow o}^1$ be the RHS of Def1, $\mathbf{A}_{\{S \rightarrow S\} \rightarrow o}^2$ be the RHS of Def6, T_1 be MON, and T_2 be ONE-BT, the top theory of ONE-BT-1. We must show $T_2 \models \mathbf{A}_{\{S \rightarrow S\} \rightarrow o}^2$. We have previously proved $(\star) T_1 \models \mathbf{A}_{\{M\} \rightarrow o}^1$. MON-to-ONE-BT = (μ, ν) is a theory morphism from T_1 to T_2 . Thus (\star) implies $T_2 \models \nu(\mathbf{A}_{\{M\} \rightarrow o}^1)$. Therefore, $T_2 \models \mathbf{A}_{\{S \rightarrow S\} \rightarrow o}^2$ since $\mathbf{A}_{\{S \rightarrow S\} \rightarrow o}^2 = \nu(\mathbf{A}_{\{M\} \rightarrow o}^1)$. \square

4. Thm20 (Thm4-via-MON-to-ONE-BT):

$$\begin{aligned} & \forall s : \{S \rightarrow S\} . \\ & \quad \text{trans-monoid } s \Rightarrow \text{MONOID}(s, \circ_{((S \rightarrow S) \times (S \rightarrow S)) \rightarrow (S \rightarrow S)} \downarrow_{s \times s}, \text{id}_{S \rightarrow S}) \\ & \quad \quad \quad (\text{transformation monoids are monoids}). \end{aligned}$$

Proof of the theorem. Similar to the proof of Thm11. \square

5. Thm28 (Thm26-via-MON-HOM-to-MON-4)

$$\begin{aligned} & \text{MON-HOM}(U_{\{M\}}, \\ & \quad U_{\{\{M\}\}}, \\ & \quad \cdot_{(M \times M) \rightarrow M}, \\ & \quad \mathbf{e}_M, \\ & \quad \cdot_{(\{M\} \times \{M\}) \rightarrow \{M\}}, \\ & \quad \mathbf{E}_{\{M\}}, \\ & \quad \mathbf{h}_{M \rightarrow \{M\}}) \quad (\text{example is a monoid homomorphism}). \end{aligned}$$

Proof of the theorem. Similar to the proof of Thm11. \square

A.5 Development of MON-ACT

1. Thm21: MON-ACTION($U_{\{M\}}, U_{\{S\}}, \cdot_{(M \times M) \rightarrow M}, \mathbf{e}_M, \mathbf{act}_{(M \times S) \rightarrow S}$)
(models of MON-ACT define monoid actions).

Proof of the theorem. Let $T = (L, \Gamma)$ be MON-ACT. We must show

$$(\star) T \models \text{MON-ACTION}(U_{\{M\}}, U_{\{S\}}, \cdot_{(M \times M) \rightarrow M}, \mathbf{e}_M, \mathbf{act}_{(M \times S) \rightarrow S}).$$

$$\Gamma \models \text{MONOID}(U_{\{M\}}, \cdot_{(M \times M) \rightarrow M}, \mathbf{e}_M) \quad (1)$$

$$\Gamma \models U_{\{S\}} \downarrow \quad (2)$$

$$\Gamma \models U_{\{S\}} \neq \emptyset_{\{S\}} \quad (3)$$

$$\Gamma \models \text{act}_{(M \times S) \rightarrow S} \downarrow (U_{\{M\}} \times U_{\{S\}}) \rightarrow U_{\{S\}} \quad (4)$$

$$\Gamma \models \forall x, y : U_{\{M\}}, s : U_{\{S\}} . x \text{ act } (y \text{ act } s) = (x \cdot y) \text{ act } s \quad (5)$$

$$\Gamma \models \forall s : U_{\{S\}} . \mathbf{e} \text{ act } s = s \quad (6)$$

$$\Gamma \models \text{MON-ACTION}(U_{\{M\}}, U_{\{S\}}, \cdot_{(M \times M) \rightarrow M}, \mathbf{e}_M, \text{act}_{(M \times S) \rightarrow S}) \quad (7)$$

(1) follows from $\text{MON} \models \text{Thm1}$ and $\text{MON} \leq T$; (2) and (3) follow from parts 1 and 2, respectively, of Lemma B.1; (4) follows from [20, Axiom 5.2] and parts 8–10 of Lemma B.1; (5) and (6) follow from Axioms 3 and 4, respectively, of T and part 5 of Lemma B.1; and (7) follows from (1)–(6) and the definition of MON-ACTION in Table 3. Therefore, (\star) holds. \square

Thm22: $\text{TOTAL}(\text{act}_{(M \times S) \rightarrow S})$ (act is total).

Proof of the theorem. Let $T = (L, \Gamma)$ be MON-ACT. $T \models \text{TOTAL}(\text{act}_{(M \times S) \rightarrow S})$ follows from Axiom 3 of T in the same way that $T \models \text{TOTAL}(\cdot_{(M \times M) \rightarrow M})$ follows from Axiom 1 of MON as shown in the proof of Thm2. \square

2. Def7: $\text{orbit}_{S \rightarrow \{S\}} = \lambda s : S . \{t : S \mid \exists x : M . x \text{ act } s = t\}$ (orbit).

Proof that RHS is defined. Similar to the proof that the RHS of Def1 is defined. \square

3. Def8: $\text{stabilizer}_{S \rightarrow \{M\}} = \lambda s : S . \{x : M \mid x \text{ act } s = s\}$ (stabilizer).

Proof that RHS is defined. Similar to the proof that the RHS of Def1 is defined. \square

4. Thm23: $\forall s : S . \text{submonoid}(\text{stabilizer } s)$ (stabilizers are submonoids).

Proof of the theorem. Let $T = (L, \Gamma)$ be MON-ACT extended by Def7 and Def8. We must show

$$(\star) T \models \forall s : S . \text{submonoid}(\text{stabilizer } s).$$

$$\Gamma \vDash (s : S) \downarrow \quad (1)$$

$$\Gamma \vDash \mathbf{e}_M \downarrow \quad (2)$$

$$\Gamma \vDash (\text{stabilizer } s) = \{x : M \mid x \text{ act } s = s\} \quad (3)$$

$$\Gamma \vDash \mathbf{e} \in (\text{stabilizer } s) \quad (4)$$

$$\Gamma \vDash (\text{stabilizer } s) \neq \emptyset_{\{M\}} \quad (5)$$

$$\begin{aligned} \Gamma \vDash \cdot \uparrow_{(\text{stabilizer } s) \times (\text{stabilizer } s)} \downarrow \\ ((\text{stabilizer } s) \times (\text{stabilizer } s)) \rightarrow (\text{stabilizer } s) \end{aligned} \quad (6)$$

$$\Gamma \vDash (\text{stabilizer } s) \downarrow \quad (7)$$

$$\begin{aligned} \Gamma \vDash \text{submonoid } (\text{stabilizer } s) = \\ (\text{stabilizer } s) \neq \emptyset_{\{M\}} \wedge \\ \cdot \uparrow_{(\text{stabilizer } s) \times (\text{stabilizer } s)} \downarrow \\ ((\text{stabilizer } s) \times (\text{stabilizer } s)) \rightarrow (\text{stabilizer } s) \wedge \\ \mathbf{e} \in (\text{stabilizer } s) \end{aligned} \quad (8)$$

$$\Gamma \vDash \text{submonoid } (\text{stabilizer } s) \quad (9)$$

$$\Gamma \vDash \forall s : S . \text{submonoid } (\text{stabilizer } s) \quad (10)$$

(1) follows from variables always being defined by [20, Axioms A5.1]; (2) follows from constants always being defined by [20, Axioms A5.2]; (3) follows from Def8 by the Equality Rules [20, Lemma A.13] and Beta-Reduction [20, Axiom A4] applied to (1) and the RHS of the result; (4) follows from (3) and Axiom 4 of T ; (5) follows immediately from (4); (6) follows from Thm2, (3), and Axiom 3 of T ; (7) follows from (3) and [20, Axiom A5.4]; (8) follows from Def1 by the Equality Rules and Beta-Reduction applied to (7) and the RHS of the result; (9) follows from (4), (5), (6), and (8) by the Tautology Rule [20, Corollary A.46]; (10) follows from (9) by Universal Generalization [20, Theorem A.30] using the fact that s is free in Γ because Γ is a set of sentences. Therefore, (\star) holds. \square

A.6 Development of ONE-BT-with-SC

1. Thm25 (Thm21-via-MON-ACT-to-ONE-BT-with-SC):

$$\begin{aligned} & \text{MON-ACTION}(\mathbb{F}_{\{S \rightarrow S\}}, \\ & \quad U_{\{S\}}, \\ & \quad \circ_{((S \rightarrow S) \times (S \rightarrow S)) \rightarrow (S \rightarrow S)} \uparrow \mathbb{F} \times \mathbb{F}, \\ & \quad \text{id}_{S \rightarrow S}, \\ & \quad \bullet_{((S \rightarrow S) \times S) \rightarrow S} \uparrow \mathbb{F} \times S) \end{aligned}$$

(second example is a monoid action).

Proof of the theorem. Similar to the proof of Thm11. \square

A.7 Development of MON-HOM

1. Thm26:

$$\begin{aligned} & \text{MON-HOM}(U_{\{M_1\}}, \\ & \quad U_{\{M_2\}}, \\ & \quad \cdot_{(M_1 \times M_1) \rightarrow M_1}, \\ & \quad \mathbf{e}_{M_1}, \\ & \quad \cdot_{(M_2 \times M_2) \rightarrow M_2}, \\ & \quad \mathbf{e}_{M_2}, \\ & \quad \mathbf{h}_{M_1 \rightarrow M_2}) \end{aligned}$$

(models of MON-HOM define monoid homomorphisms).

Proof of the theorem. Let $T = (L, \Gamma)$ be MON-HOM and \mathbf{A}_o be

$$\begin{aligned} & \text{MON-HOM}(U_{\{M_1\}}, U_{\{M_2\}}, \cdot_{(M_1 \times M_1) \rightarrow M_1}, \mathbf{e}_{M_1}, \cdot_{(M_2 \times M_2) \rightarrow M_2}, \\ & \quad \mathbf{e}_{M_2}, \mathbf{h}_{M_1 \rightarrow M_2}). \end{aligned}$$

We must show $(\star) T \models \mathbf{A}_o$.

$$\Gamma \models \text{MONOID}(U_{\{M_1\}}, \cdot_{(M_1 \times M_1) \rightarrow M_1}, \mathbf{e}_{M_1}) \quad (1)$$

$$\Gamma \models \text{MONOID}(U_{\{M_2\}}, \cdot_{(M_2 \times M_2) \rightarrow M_2}, \mathbf{e}_{M_2}) \quad (2)$$

$$\Gamma \models \mathbf{h}_{M_1 \rightarrow M_2} \downarrow U_{\{M_1\}} \rightarrow U_{\{M_2\}} \quad (3)$$

$$\Gamma \models \forall x, y : U_{\{M_1\}} \cdot \mathbf{h}(x \cdot y) = (\mathbf{h}x) \cdot (\mathbf{h}y) \quad (4)$$

$$\Gamma \models \mathbf{A}_o \quad (5)$$

(1) and (2) follow similarly to the proof of Thm1; (3) follows from [20, Axiom 5.2] and parts 8 and 9 of Lemma B.1; (4) follows from Axiom 5 of T and part 5 of Lemma B.1; (5) follows from (1)–(4), Axiom 6 of T , and the definition of MON-HOM in Table 3. Therefore, (\star) holds. \square

Thm27: $\text{TOTAL}(\mathbf{h}_{M_1 \rightarrow M_2})$ ($\mathbf{h}_{M_1 \rightarrow M_2}$ is total).

Proof of the theorem. Let $T = (L, \Gamma)$ be MON-HOM. $T \models \text{TOTAL}(\mathbf{h}_{M_1 \rightarrow M_2})$ follows from Axiom 5 of T in the same way that $T \models \text{TOTAL}(\cdot_{(M \times M) \rightarrow M})$ follows from Axiom 1 of MON as shown in the proof of Thm2. \square

A.8 Development of MON-over-COF

1. Def9: $\text{prod}_{R \rightarrow R \rightarrow (R \rightarrow M) \rightarrow M} =$
 $\Gamma f : Z_{\{R\}} \rightarrow Z_{\{R\}} \rightarrow (Z_{\{R\}} \rightarrow M) \rightarrow M .$
 $\forall m, n : Z_{\{R\}}, g : Z_{\{R\}} \rightarrow M . f m n g \simeq$
 $(m > n \mapsto e \mid (f m (n - 1) g) \cdot (g n))$ (iterated product).

Proof that RHS is defined. Let

$$\mathbf{A}_o = \forall m, n : Z_{\{R\}}, g : Z_{\{R\}} \rightarrow M . f m n g \simeq$$

$$(m > n \mapsto e \mid (f m (n - 1) g) \cdot (g n)).$$

Suppose that two functions f_1 and f_2 satisfy \mathbf{A}_o . It is easy to see that f_1 and f_2 must be the same function based on the recursive structure of f in \mathbf{A}_o . Thus, \mathbf{A}_o specifies a unique function, and so the RHS of Def9 is defined by [20, Axiom A6.1]. \square

2. Thm29: $\forall m : Z_{\{R\}}, g : Z_{\{R\}} \rightarrow M . \left(\prod_{i=m}^m g i \right) \simeq g m$
(trivial product).

Proof of the theorem. Let \mathbf{A}_o be the theorem and $T = (L, \Gamma)$ be COM-MON-over-COF extended with Def9. We must show $(\star) T \models \mathbf{A}_o$.

Let Δ be the set $\{m \in Z_{\{R\}}, g \in Z_{\{R\}} \rightarrow M\}$.

$$\Gamma \cup \Delta \models (m : R) \downarrow \wedge (g : R \rightarrow M) \downarrow \tag{1}$$

$$\Gamma \cup \Delta \models \left(\prod_{i=m}^m g i \right) \simeq \left(\prod_{i=m}^{m-1} g i \right) \cdot g m \tag{2}$$

$$\Gamma \cup \Delta \models \left(\prod_{i=m}^{m-1} g i \right) \cdot g m \simeq e \cdot g m \tag{3}$$

$$\Gamma \cup \Delta \models e \cdot g m \simeq g m \tag{4}$$

$$\Gamma \cup \Delta \models \left(\prod_{i=m}^m g i \right) \simeq g m \tag{5}$$

$$\Gamma \models \mathbf{A}_o \tag{6}$$

(1) follows from variables always being defined by [20, Axiom A5.1]; (2) and (3) follow from (1) and Def9; (4) follows from Axiom 20 of T ; (5) follows from (2), (3), and (4) by the Quasi-Equality Rules [20, Lemma A.4]; and (6) follows from (5) by the Deduction Theorem [20, Theorem A.50] and by Universal Generalization [20, Theorem A.30]

using the fact that m and g are not free in Γ since Γ is a set of sentences. Therefore, (\star) holds. \square

3. Thm30: $\forall m, k, n : Z_{\{R\}}, g : Z_{\{R\}} \rightarrow M$.

$$m < k < n \Rightarrow \left(\prod_{i=m}^k g i \right) \cdot \left(\prod_{i=k+1}^n g i \right) \simeq \prod_{i=m}^n g i$$

(extended iterated product).

Proof of the theorem. Let \mathbf{A}_o be the theorem and $T = (L, \Gamma)$ be MON-over-COF extended by Def9. We must show (a) $T \models \mathbf{A}_o$.

Let Δ be the set

$$\{m \in Z_{\{R\}}, k \in Z_{\{R\}}, n \in Z_{\{R\}}, m < k < n\}.$$

We will prove

$$(b) \Gamma \cup \Delta \models \left(\prod_{i=m}^k g i \right) \cdot \left(\prod_{i=k+1}^n g i \right) \simeq \left(\prod_{i=m}^n g i \right)$$

from all $n > k$ by induction on the n .

Base case: $n = k + 1$. Then:

$$\Gamma \cup \Delta \models (m : R) \downarrow \wedge (k : R) \downarrow \wedge (n : R) \downarrow \wedge (g : R \rightarrow M) \downarrow \quad (1)$$

$$\Gamma \cup \Delta \models \left(\prod_{i=m}^k g i \right) \cdot \left(\prod_{i=k+1}^n g i \right) \simeq \left(\prod_{i=m}^k g i \right) \cdot g n \quad (2)$$

$$\Gamma \cup \Delta \models \left(\prod_{i=m}^k g i \right) \cdot g n \simeq \left(\prod_{i=m}^n g i \right) \quad (3)$$

(1) follows from variables always being defined by [20, Axiom A5.1]; (2) follows from $n = k + 1$ and Thm29; and (3) follows from $n = k + 1$, (1), and Def9. Thus (b) holds by the Quasi-Equality Rules [20, Lemma A.4] when $n = k + 1$.

Induction step: $n > k + 1$ and assume

$$\Gamma \cup \Delta \models \left(\prod_{i=m}^k g i \right) \cdot \left(\prod_{i=k+1}^{n-1} g i \right) \simeq \left(\prod_{i=m}^{n-1} g i \right).$$

Then:

$$\Gamma \cup \Delta \vDash (m : R)\downarrow \wedge (k : R)\downarrow \wedge (n : R)\downarrow \wedge (g : R \rightarrow M)\downarrow \quad (1)$$

$$\Gamma \cup \Delta \vDash \left(\prod_{i=m}^k g i \right) \cdot \left(\prod_{i=k+1}^n g i \right) \simeq \left(\prod_{i=m}^k g i \right) \cdot \left(\left(\prod_{i=k+1}^{n-1} g i \right) \cdot g n \right) \quad (2)$$

$$\Gamma \cup \Delta \vDash \left(\prod_{i=m}^k g i \right) \cdot \left(\left(\prod_{i=k+1}^{n-1} g i \right) \cdot g n \right) \simeq \left(\prod_{i=m}^{n-1} g i \right) \cdot g n \quad (3)$$

$$\Gamma \cup \Delta \vDash \left(\prod_{i=m}^{n-1} g i \right) \cdot g n \simeq \left(\prod_{i=m}^n g i \right) \quad (4)$$

(1) follows from variables always being defined by [20, Axiom A5.1]; (2) and (4) follows from (1) and Def9; and (3) follows from Axiom 19 of T and the induction hypothesis. Thus (b) holds by the Quasi-Equality Rules [20, Lemma A.4] when $n > k + 1$.

Therefore, (b) holds for all $n > k$, and (a) follows from this by the Deduction Theorem [20, Theorem A.50] and by Universal Generalization [20, Theorem A.30] using the fact that m , k , n , and g are not free in Γ since Γ is a set of sentences. \square

A.9 Development of COM-MON-over-COF

1. Thm31: $\forall m, n : Z_{\{R\}}, g, h : Z_{\{R\}} \rightarrow M$.

$$\left(\prod_{i=m}^n g i \right) \cdot \left(\prod_{i=m}^n h i \right) \simeq \prod_{i=m}^n (g i) \cdot (h i)$$

(product of iterated products).

Proof of the theorem. Let \mathbf{A}_o be the theorem and $T = (L, \Gamma)$ be COM-MON-over-COF extended by Def9. We must show (a) $T \vDash \mathbf{A}_o$.

Let Δ be the set $\{n \in Z_{\{R\}}, g \in Z_{\{R\}} \rightarrow M\}$. We will prove

$$(b) \Gamma \cup \Delta \vDash \left(\prod_{i=m}^n g i \right) \cdot \left(\prod_{i=m}^n h i \right) \simeq \prod_{i=m}^n (g i) \cdot (h i)$$

for all n by induction on the n .

Base case: $n < m$. Then:

$$\Gamma \cup \Delta \vDash (n : R)\downarrow \wedge (g : R \rightarrow M)\downarrow \quad (1)$$

$$\Gamma \cup \Delta \vDash \left(\prod_{i=m}^n g i \right) \cdot \left(\prod_{i=m}^n h i \right) \simeq \mathbf{e} \cdot \mathbf{e} \quad (2)$$

$$\Gamma \cup \Delta \vDash \prod_{i=m}^n (g i) \cdot (h i) \simeq \mathbf{e} \quad (3)$$

(1) follows from variables always being defined by [20, Axiom A5.1]; and (2) and (3) follow from $n < m$, (1), and Def9. Thus (b) holds by Axiom 20 of T and the Quasi-Equality Rules [20, Lemma A.4] when $n < m$.

Induction step: $n \geq m$ and assume

$$\Gamma \cup \Delta \vDash \left(\prod_{i=m}^{n-1} g i \right) \cdot \left(\prod_{i=m}^{n-1} h i \right) \simeq \prod_{i=m}^{n-1} (g i) \cdot (h i).$$

Then:

$$\Gamma \cup \Delta \vDash (n : R)\downarrow \wedge (g : R \rightarrow M)\downarrow \quad (1)$$

$$\Gamma \cup \Delta \vDash \left(\prod_{i=m}^n g i \right) \cdot \left(\prod_{i=m}^n h i \right) \simeq \left(\prod_{i=m}^{n-1} g i \right) \cdot g n \cdot \left(\prod_{i=m}^{n-1} h i \right) \cdot h n \quad (2)$$

$$\begin{aligned} \Gamma \cup \Delta \vDash \left(\prod_{i=m}^{n-1} g i \right) \cdot g n \cdot \left(\prod_{i=m}^{n-1} h i \right) \cdot h n &\simeq \\ \left(\prod_{i=m}^{n-1} g i \right) \cdot \left(\prod_{i=m}^{n-1} h i \right) \cdot g n \cdot h n &\quad (3) \end{aligned}$$

$$\begin{aligned} \Gamma \cup \Delta \vDash \left(\prod_{i=m}^{n-1} g i \right) \cdot \left(\prod_{i=m}^{n-1} h i \right) \cdot g n \cdot h n &\simeq \\ \left(\prod_{i=m}^{n-1} (g i) \cdot (h i) \right) \cdot (g n \cdot h n) &\quad (4) \end{aligned}$$

$$\Gamma \cup \Delta \vDash \left(\prod_{i=m}^{n-1} (g i) \cdot (h i) \right) \cdot (g n \cdot h n) \simeq \prod_{i=m}^n (g i) \cdot (h i) \quad (5)$$

(1) follows from variables always being defined by [20, Axiom A5.1]; (2) and (5) follow from (1) and Def9; (3) follows from Axiom 21 of T ; and (4) follows from the induction hypothesis. Thus (b) holds by the Quasi-Equality Rules [20, Lemma A.4] when $n \geq m$.

Therefore, (b) holds for all n , and (a) follows from this by the Deduction Theorem [20, Theorem A.50] and by Universal Generalization [20, Theorem A.30] using the fact that n and g are not free in Γ since Γ is a set of sentences. \square

A.10 Development of COM-MON-ACT-over-COF

1. Thm32: $\forall x, y : M, s : S . x \text{ act } (y \text{ act } s) = y \text{ act } (x \text{ act } s)$
(act has commutative-like property).

Proof of the theorem. Let \mathbf{A}_o be the theorem and $T = (L, \Gamma)$ be COM-MON-ACT-over-COF. We must show $(\star) T \models \mathbf{A}_o$.

$$\Gamma \models (x : M)\downarrow \wedge (y : M)\downarrow \wedge (s : S)\downarrow \tag{1}$$

$$\Gamma \models x \text{ act } (y \text{ act } s) = (x \cdot y) \text{ act } s \tag{2}$$

$$\Gamma \models y \text{ act } (x \text{ act } s) = (y \cdot x) \text{ act } s \tag{3}$$

$$\Gamma \models x \cdot y = y \cdot x \tag{4}$$

$$\Gamma \models y \text{ act } (x \text{ act } s) = (x \cdot y) \text{ act } s \tag{5}$$

$$\Gamma \models x \text{ act } (y \text{ act } s) = y \text{ act } (x \text{ act } s) \tag{6}$$

$$\Gamma \models \mathbf{A}_o \tag{7}$$

(1) follows from variables always being defined by [20, Axiom A5.1]; (2) and (3) follow from (1) and Axiom 22 of T by Universal Instantiation [20, Theorem A.14]; (4) follows from (1) and Axiom 21 of T by Universal Instantiation; (5) follows from (4) and (3) by Quasi-Equality Substitution [20, Lemma A.2]; (6) follows from (2) and (5) by the Equality Rules [20, Lemma A.13]; (7) follows from (6) by Universal Generalization [20, Theorem A.30] using the fact that x , y , and s are not free in Γ since Γ is a set of sentences. Therefore, (\star) holds. \square

A.11 Development of STR

1. Def10: $\text{str}_{\{R \rightarrow A\}} = [A]$ (string quasitype).

Proof that RHS is defined. Let T be the top theory of STR-1. We must show $(\star) T \models [A]\downarrow$. Now $[A]$ stands for

$$\{s : \langle\langle A \rangle\rangle \mid \exists n : \mathbf{C}_{\{R\}}^N . \forall m : \mathbf{C}_{\{R\}}^N . (s m)\downarrow \Leftrightarrow \mathbf{C}_{A \rightarrow A \rightarrow o} m n\}$$

based on the notational definitions in Table 10.1 in [20]. Thus (\star) holds because function abstractions are always defined by [20, Axiom A5.11]. \square

2. Def11: $\epsilon_{R \rightarrow A} = []_{R \rightarrow A}$ (empty string).

Proof that RHS is defined. Let T be the top theory of STR-1. We must show $(\star) T \models []_{R \rightarrow A} \downarrow$. Now $[]_{R \rightarrow A}$ stands for

$$\lambda x : R . \perp_A$$

based on the notational definitions in Tables 6.4 and 10.1 in [20]. Thus (\star) holds because function abstractions are always defined by [20, Axiom A5.11]. \square

Def12: $\text{cat}_{((R \rightarrow A) \times (R \rightarrow A)) \rightarrow (R \rightarrow A)} =$

$$\text{I } f : (\text{str} \times \text{str}) \rightarrow \text{str} .$$

$$\forall x : \text{str} . f(\epsilon, x) = x \wedge$$

$$\forall a : A, x, y : \text{str} . f(a :: x, y) = a :: f(x, y). \quad (\text{concatenation}).$$

Proof that RHS is defined. Let T be the top theory of STR-1 extended by Def10 and Def11. We must show

$$(\star) T \models \text{cat}_{((R \rightarrow A) \times (R \rightarrow A)) \rightarrow (R \rightarrow A)} \downarrow.$$

The body of the RHS of the definition uniquely specifies the concatenation function of members of str . Thus (\star) holds by [20, Axioms A5.8 and A6.1]. \square

3. Thm33: $\forall x : \text{str} . \epsilon x = x \epsilon = x$ (ϵ is an identity element).

Proof of the theorem. Let $T = (L, \Gamma)$ be the top theory of STR-1 extended by Def10–Def12. We must show:

$$(a) T \models \forall x : \text{str} . \epsilon x = x.$$

$$(b) T \models \forall x : \text{str} . x \epsilon = x.$$

Let Δ be the set $\{x \in \text{str}\}$. Then:

$$\Gamma \cup \Delta \models \epsilon x = x \tag{1}$$

$$\Gamma \models \forall x : \text{str} . \epsilon x = x \tag{2}$$

(1) follows from $x \in \text{str}$ and Def12; and (2) follows from (1) by the Deduction Theorem [20, Theorem A.50] and then by Universal Generalization [20, Theorem A.30] using the fact that $(x : R \rightarrow A)$ is not free in Γ since Γ is a set of sentences. Therefore, (a) holds.

We will prove (c) $\Gamma \cup \Delta \vDash x\epsilon = x$ by induction on the length of x .

Base case: x is ϵ . Then $\Gamma \cup \Delta \vDash \epsilon\epsilon = \epsilon$ is an instance of (1) above.

Induction step: x is $(a :: y)$ and assume $\Gamma \cup \Delta \vDash y\epsilon = y$. Then:

$$\Gamma \cup \Delta \vDash (a :: y)\epsilon = (a :: y\epsilon) \quad (1)$$

$$\Gamma \cup \Delta \vDash (a :: y)\epsilon = (a :: y) \quad (2)$$

(1) follows from $x \in \text{str}$ and Def12; and (2) follows from the induction hypothesis and (1) by Quasi-Equality Substitution [20, Lemma A.2]. Thus $\Gamma \cup \Delta \vDash (a :: y)\epsilon = (a :: y)$ holds by the Equality Rules [20, Lemma A.13].

Therefore (c) holds, and (b) follows from (c) by the Deduction Theorem [20, Theorem A.50] and then by Universal Generalization [20, Theorem A.30] using the fact that $(x : R \rightarrow A)$ is not free in Γ since Γ is a set of sentences. \square

4. Thm34: $\forall x, y, z : \text{str} . x(yz) = (xy)z$ (cat is associative).

Proof of the theorem. Let $T = (L, \Gamma)$ be the top theory of STR-1 extended by Def10–Def12. We must show

$$(a) T \vDash \forall x, y, z : \text{str} . x(yz) = (xy)z.$$

Let Δ be the set $\{x \in \text{str}, y \in \text{str}, z \in \text{str}\}$. We will prove

$$(b) \Gamma \cup \Delta \vDash x(yz) = (xy)z$$

by induction on the length of x .

Base case: x is ϵ . Then:

$$\Gamma \cup \Delta \vDash \epsilon(yz) = (yz) \quad (1)$$

$$\Gamma \cup \Delta \vDash (yz) = (\epsilon y)z \quad (2)$$

(1) and (2) follow from Thm33. Thus $\Gamma \cup \Delta \vDash \epsilon(yz) = (\epsilon y)z$ holds by the Equality Rules [20, Lemma A.13].

Induction step: x is $(a :: w)$ and assume $\Gamma \cup \Delta \vDash w(yz) = (wy)z$. Then:

$$\Gamma \cup \Delta \vDash (a :: w)(yz) = a :: w(yz) \quad (1)$$

$$\Gamma \cup \Delta \vDash a :: w(yz) = a :: (wy)z \quad (2)$$

$$\Gamma \cup \Delta \vDash a :: (wy)z = (a :: wy)z \quad (3)$$

$$\Gamma \cup \Delta \vDash (a :: wy)z = ((a :: w)y)z \quad (4)$$

(1), (3), and (4) follow from $x \in \mathbf{str}$, $y \in \mathbf{str}$, and $z \in \mathbf{str}$ and Def12; and (2) follows from the induction hypothesis. Thus

$$\Gamma \cup \Delta \vDash (a :: w)(yz) = ((a :: w)y)z$$

holds by the Equality Rules [20, Lemma A.13].

Therefore (b) holds, and (a) follows from (b) by the Deduction Theorem [20, Theorem A.50] and then by Universal Generalization [20, Theorem A.30] using the fact that $(x : R \rightarrow A)$, $(y : R \rightarrow A)$, and $(z : R \rightarrow A)$ are not free in Γ since Γ is a set of sentences. \square

5. Thm35 (Thm1-via-MON-over-COF-to-STR-2):

$$\text{MONOID}(\mathbf{str}_{\{R \rightarrow A\}}, \text{cat}_{((R \rightarrow A) \times (R \rightarrow A)) \rightarrow (R \rightarrow A)}, \epsilon_{R \rightarrow A})$$

(strings form a monoid).

Proof of the theorem. Similar to the proof of Thm11. \square

6. Def13 (Def3-via-MON-over-COF-to-STR-2):

$$\begin{aligned} & \text{set-cat}_{(\{R \rightarrow A\} \times \{R \rightarrow A\}) \rightarrow \{R \rightarrow A\}} = \\ & \text{set-op}_{(((R \rightarrow A) \times (R \rightarrow A)) \rightarrow (R \rightarrow A)) \rightarrow ((\{R \rightarrow A\} \times \{R \rightarrow A\}) \rightarrow \{R \rightarrow A\})} \text{cat} \end{aligned}$$

(set concatenation).

Proof that RHS is defined. Similar to the proof that the RHS of Def6 is defined. \square

7. Def14 (Def4-via-MON-over-COF-to-STR-2):

$$E_{\{R \rightarrow A\}} = \{\epsilon_{R \rightarrow A}\} \quad (\text{set identity element}).$$

Proof that RHS is defined. Similar to the proof that the RHS of Def6 is defined. \square

8. Thm36 (Thm12-via-MON-over-COF-1-to-STR-2):

$$\text{MONOID}(\mathcal{P}(\mathbf{str}_{\{R \rightarrow A\}}), \text{set-cat}_{(\{R \rightarrow A\} \times \{R \rightarrow A\}) \rightarrow \{R \rightarrow A\}}, E_{\{R \rightarrow A\}})$$

(string sets form a monoid).

Proof of the theorem. Similar to the proof of Thm11. \square

9. Def15 (Def9-via-MON-over-COF-1-to-STR-2):

$$\begin{aligned} & \text{iter-cat}_{R \rightarrow R \rightarrow (R \rightarrow (R \rightarrow A)) \rightarrow (R \rightarrow A)} = \\ & \text{I } f : Z_{\{R\}} \rightarrow Z_{\{R\}} \rightarrow (Z_{\{R\}} \rightarrow (R \rightarrow A)) \rightarrow (R \rightarrow A) . \\ & \quad \forall m, n : Z_{\{R\}}, g : Z_{\{R\}} \rightarrow (R \rightarrow A) . f m n g \simeq \\ & \quad (m > n \mapsto \epsilon \mid (f m (n - 1) g) \text{ cat } (g n)) \end{aligned}$$

(iterated concatenation).

Proof that RHS is defined. Similar to the proof that the RHS of Def6 is defined. \square

B Miscellaneous Theorems

Lemma B.1 (Universal Sets) *The following formulas are valid:*

1. $U_{\{\alpha\}} \downarrow$.
2. $U_{\{\alpha\}} \neq \emptyset_{\{\alpha\}}$.
3. $\forall x : \alpha . x \in U_{\{\alpha\}}$.
4. $(\lambda \mathbf{x} : \alpha . \mathbf{B}_\beta) \Leftrightarrow (\lambda \mathbf{x} : U_{\{\alpha\}} . \mathbf{B}_\beta)$.
5. $(\forall \mathbf{x} : \alpha . \mathbf{B}_o) \Leftrightarrow (\forall \mathbf{x} : U_{\{\alpha\}} . \mathbf{B}_o)$.
6. $(\exists \mathbf{x} : \alpha . \mathbf{B}_o) \Leftrightarrow (\exists \mathbf{x} : U_{\{\alpha\}} . \mathbf{B}_o)$.
7. $(\mathbf{I} \mathbf{x} : \alpha . \mathbf{B}_o) \Leftrightarrow (\mathbf{I} \mathbf{x} : U_{\{\alpha\}} . \mathbf{B}_o)$.
8. $\mathbf{A}_\alpha \downarrow \Leftrightarrow (\mathbf{A}_\alpha \downarrow U_{\{\alpha\}})$
9. $U_{\{\alpha \rightarrow \beta\}} = (U_{\{\alpha\}} \rightarrow U_{\{\beta\}})$.
10. $U_{\{\alpha \times \beta\}} = (U_{\{\alpha\}} \times U_{\{\beta\}})$.
11. $U_{\{\{\alpha\}\}} = \mathcal{P}(U_{\{\alpha\}})$.
12. $\mathbf{A}_{(\alpha \times \beta) \rightarrow \gamma} = \mathbf{A}_{(\alpha \times \beta) \rightarrow \gamma} \uparrow U_{\{\alpha\}} \times U_{\{\beta\}}$.

Proof The proof is left to the reader as an exercise. □

The following lemma is an expanded version of [20, Lemma 13.9]:

Lemma B.2 *Let $\Phi = (\mu, \nu)$ be a translation from T_1 to T_2 .*

1. *If $\alpha \in \mathcal{T}(L_1)$ and $\bar{\mu}(\alpha) \in \mathcal{Q}_2$, then $T_2 \models \bar{\mu}(\alpha) \downarrow \Rightarrow \bar{\nu}(U_{\{\alpha\}}) = \bar{\mu}(\alpha)$.*
2. *If $\alpha \in \mathcal{T}(L_1)$ and $\bar{\mu}(\alpha) \in \mathcal{Q}_2$, then $T_2 \models \bar{\mu}(\alpha) \uparrow \Rightarrow \bar{\nu}(U_{\{\alpha\}}) = \emptyset_{\{\tau(\bar{\mu}(\alpha))\}}$.*
3. *If $\mathbf{a} \in \mathcal{B}_1$ and $\mu(\mathbf{a}) \in \mathcal{Q}_2$, then*

$$T_2 \models \bar{\nu}(U_{\{\mathbf{a}\}} \neq \emptyset_{\{\mathbf{a}\}}) \Leftrightarrow (\mu(\mathbf{a}) \downarrow \wedge \mu(\mathbf{a}) \neq \emptyset_{\{\tau(\mu(\mathbf{a}))\}}).$$

4. *If $\mathbf{c}_\alpha \in \mathcal{C}_1$ and $\bar{\mu}(\alpha) \in \mathcal{T}_2$, then $T_2 \models \bar{\nu}(\mathbf{c}_\alpha \downarrow U_{\{\alpha\}}) \Leftrightarrow \nu(\mathbf{c}_\alpha) \downarrow$.*
5. *If $\mathbf{c}_\alpha \in \mathcal{C}_1$ and $\bar{\mu}(\alpha) \in \mathcal{Q}_2$, then*

$$T_2 \models \bar{\mu}(\alpha) \downarrow \Rightarrow \bar{\nu}(\mathbf{c}_\alpha \downarrow U_{\{\alpha\}}) \Leftrightarrow \nu(\mathbf{c}_\alpha) \downarrow \bar{\mu}(\alpha).$$

Proof Let $\alpha \in \mathcal{T}(L_1)$, $\bar{\mu}(\alpha) \in \mathcal{Q}_2$, and $(\star) M$ be a general model of T_2 in which $\bar{\mu}(\alpha)\downarrow$ is true. We must show that $\bar{\nu}(U_{\{\alpha\}}) = \bar{\mu}(\alpha)$ is true in M . Then

$$\begin{aligned} & \bar{\nu}(U_{\{\alpha\}}) \\ \equiv & \bar{\nu}(\lambda x : \alpha . T_o) \\ \equiv & \lambda x : \bar{\mu}(\alpha) . T_o \\ \equiv & \lambda x : \tau(\bar{\mu}(\alpha)) . (x \in \bar{\mu}(\alpha) \mapsto T_o \mid F_o). \end{aligned}$$

by the definition of $\bar{\nu}$ and notational definitions. The last expression is clearly equal to $\lambda x : \tau(\bar{\mu}(\alpha)) . \bar{\mu}(\alpha) x$, which is equal to $\bar{\mu}(\alpha)$ in M by (\star) . This proves part 1. Part 2 follows from the proof of part 1 and the notational definition for the empty set pseudoconstant. Part 3 follows immediately from the definition of $\bar{\nu}$ and parts 1 and 2. Part 4 follows from the definition of $\bar{\nu}$ and the notational definition for the defined-in-quasitype operator. And part 5 follows immediately from the definition of $\bar{\nu}$ and part 1. \square

Theorem B.3 (Existential Instantiation) *If $\Gamma \vdash_{\mathfrak{A}} \mathbf{A}_o \Rightarrow \mathbf{B}_o$, then*

$$\Gamma \vdash_{\mathfrak{A}} (\exists \mathbf{x} : \alpha . \mathbf{A}_o) \Rightarrow \mathbf{B}_o,$$

provided that $(\mathbf{x} : \alpha)$ is not free in \mathbf{B}_o or in any member of Γ .

Proof Let $(\star) (\mathbf{x} : \alpha)$ be not free in \mathbf{B}_o or in any member of Γ .

$$\Gamma \vdash_{\mathfrak{A}} \mathbf{A}_o \Rightarrow \mathbf{B}_o \tag{1}$$

$$\Gamma \vdash_{\mathfrak{A}} \neg \mathbf{B}_o \Rightarrow \neg \mathbf{A}_o \tag{2}$$

$$\Gamma \vdash_{\mathfrak{A}} \neg \mathbf{B}_o \Rightarrow \forall \mathbf{x} : \alpha . \neg \mathbf{A}_o \tag{3}$$

$$\Gamma \vdash_{\mathfrak{A}} \neg \mathbf{B}_o \Rightarrow \neg \neg \forall \mathbf{x} : \alpha . \neg \mathbf{A}_o \tag{4}$$

$$\Gamma \vdash_{\mathfrak{A}} \neg \mathbf{B}_o \Rightarrow \neg \exists \mathbf{x} : \alpha . \mathbf{A}_o \tag{5}$$

$$\Gamma \vdash_{\mathfrak{A}} (\exists \mathbf{x} : \alpha . \mathbf{A}_o) \Rightarrow \mathbf{B}_o \tag{6}$$

(1) is given; (2) follows from (1), (4) follows from (3), and (6) follows from (5) by the Tautology Rule [20, Corollary A.46]; (3) follows from (2) and (\star) by [20, Lemma A.47] and (5) follows from (4) by the notational definition for \exists . \square

Lemma B.4 *Let T be MON extended by the definition Def2. The formula*

$$\mathbf{A}_M \cdot^{\text{op}} \mathbf{B}_M \simeq \mathbf{B}_M \cdot \mathbf{A}_M$$

is valid in T .

Proof Let \mathbf{X}_o be

$$\mathbf{A}_M \cdot^{\text{op}} \mathbf{B}_M \simeq \mathbf{B}_M \cdot \mathbf{A}_M,$$

N be a model of T , and $\varphi \in \text{assign}(N)$. Suppose that $V_\varphi^N(\mathbf{A}_M)$ or $V_\varphi^N(\mathbf{B}_M)$ is undefined. Then clearly $V_\varphi^N(\mathbf{X}_o) = \top$. Now suppose that $V_\varphi^N(\mathbf{A}_M)$ and $V_\varphi^N(\mathbf{B}_M)$ are defined. Then $V_\varphi^N(\mathbf{X}_o) = \top$ by Def2. \square

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