

# Simple Type Theory with Undefinedness, Quotation, and Evaluation<sup>\*</sup>

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## Abstract

This paper presents a version of simple type theory called  $\mathcal{Q}_0^{\text{uqe}}$  that is based on  $\mathcal{Q}_0$ , the elegant formulation of Church's type theory created and extensively studied by Peter B. Andrews.  $\mathcal{Q}_0^{\text{uqe}}$  directly formalizes the *traditional approach to undefinedness* in which undefined expressions are treated as legitimate, nondenoting expressions that can be components of meaningful statements.  $\mathcal{Q}_0^{\text{uqe}}$  is also equipped with a facility for reasoning about the syntax of expressions based on *quotation* and *evaluation*. Quotation is used to refer to a syntactic value that represents the syntactic structure of an expression, and evaluation is used to refer to the value of the expression that a syntactic value represents. With quotation and evaluation it is possible to reason in  $\mathcal{Q}_0^{\text{uqe}}$  about the interplay of the syntax and semantics of expressions and, as a result, to formalize in  $\mathcal{Q}_0^{\text{uqe}}$  syntax-based mathematical algorithms. The paper gives the syntax and semantics of  $\mathcal{Q}_0^{\text{uqe}}$  as well as a proof system for  $\mathcal{Q}_0^{\text{uqe}}$ . The proof system is shown to be sound for all formulas and complete for formulas that do not contain evaluations. The paper also illustrates some applications of  $\mathcal{Q}_0^{\text{uqe}}$ .

**Keywords:** Church's type theory, undefinedness, reasoning about syntax, quotation, evaluation, truth predicates, substitution.

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# 1 Introduction

A huge portion of mathematical reasoning is performed by algorithmically manipulating the syntactic structure of mathematical expressions. For example, the derivative of a function is commonly obtained using an algorithm that repeatedly applies syntactic differentiation rules to an expression that represents the function. The specification and analysis of a syntax-based mathematical algorithm requires the ability to reason about the interplay of how the expressions are manipulated and what the manipulations mean mathematically. This is challenging to do in a traditional logic like first-order logic or simple type theory because there is no mechanism for directly referring to the syntax of the expressions in the logic.

The standard approach for reasoning in a logic about a language  $L$  of expressions is to introduce another language  $L_{\text{syn}}$  to represent the syntax of  $L$ . The expressions in  $L_{\text{syn}}$  denote certain *syntactic values* (e.g., syntax trees) that represent the syntactic structures of the expressions in  $L$ . We will thus call  $L_{\text{syn}}$  a *syntax language*. A syntax language like  $L_{\text{syn}}$  is usually presented as an *inductive type*. The members of  $L$  are mapped by a *quotation function* to members of  $L_{\text{syn}}$ , and members of  $L_{\text{syn}}$  are mapped by an *evaluation function* to members of  $L$ . The language  $L_{\text{syn}}$  provides the means to indirectly reason about the members of  $L$  as syntactic objects, and the quotation and evaluation functions link this reasoning directly to  $L$  itself. In computer science this approach is called a *deep embedding* [8]. The components of the standard approach — a syntax language, quotation function, and evaluation function — form an instance of a *syntax framework* [28], a mathematical structure that models systems for reasoning about the syntax of an interpreted language.

We will say that an implementation of the standard approach is *global* when  $L$  is the entire language of the logic and is *local* otherwise. For example, the use of Gödel numbers to represent the syntactic structure of expressions is usually a global approach since every expression is assigned a Gödel number. We will also say that an implementation of the standard approach is *internal* when the quotation and evaluation functions are expressed as operators in the logic and is *external* when they are expressed only in the metalogic. Let the *replete approach* be the standard approach restricted to implementations that are both global and internal. The components of an implementation of the replete approach form an instance of a *replete syntax framework* [28].

It is a straightforward task to implement the local approach in a traditional logic, but two significant shortcomings cannot be easily avoided. First, the implementation must be external since the quotation function, and often the evaluation function as well, can only be expressed in the metalogic, not in the logic itself. Second, the constructed syntax framework works only for  $L$ ; another language (e.g., a larger language that includes  $L$ ) requires a new syntax framework. For instance, each time a defined constant is added to  $L$ , the syntax language, quotation function, and evaluation function must all be extended. See [25] for a more detailed presentation of the local approach.

Implementing the replete approach is much more ambitious: quotation and

evaluation operators are added to the logic and then a syntax framework is built for the entire language of the logic. We will write the quotation and evaluation operators applied to an expression  $e$  as  $\ulcorner e \urcorner$  and  $\llbracket e \rrbracket$ , respectively. The replete approach provides the means to directly reason about the syntax of the entire language of the logic in the logic itself. Moreover, the syntax framework does not have to be extended whenever the language of the logic is extended, and it can be used to express syntactic side conditions, schemas, substitution operations, and other such things directly in the logic. In short, the replete approach enables syntax-based reasoning to be moved from the metalogic to the logic itself.

At first glance, the replete approach appears to solve the whole problem of how to reason about the interplay of syntax and semantics. However, the replete approach comes with an entourage of challenging problems that stand in the way of an effective implementation. Of these, we are most concerned with the following two:

1. *Evaluation Problem.* Since a replete syntax framework works for the entire language of the logic, the evaluation operator is applicable to formulas and thus is effectively a truth predicate. Hence, by the proof of Alfred Tarski's theorem on the undefinability of truth [62, 63, 64], if the evaluation operator is total in the context of a sufficiently strong theory like first-order Peano arithmetic, then it is possible to express the liar paradox using the quotation and evaluation operators. Therefore, the evaluation operator must be partial and the law of disquotation cannot hold universally (i.e., for some expressions  $e$ ,  $\llbracket \ulcorner e \urcorner \rrbracket \neq e$ ). As a result, reasoning with evaluation is cumbersome and leads to undefined expressions.
2. *Variable Problem.* The variable  $x$  is not free in the expression  $\ulcorner x + 3 \urcorner$  (or in any quotation). However,  $x$  is free in  $\llbracket \ulcorner x + 3 \urcorner \rrbracket$  because  $\llbracket \ulcorner x + 3 \urcorner \rrbracket = x + 3$ . If the value of a constant  $c$  is  $\ulcorner x + 3 \urcorner$ , then  $x$  is free in  $\llbracket c \rrbracket$  because  $\llbracket c \rrbracket = \llbracket \ulcorner x + 3 \urcorner \rrbracket = x + 3$ . Hence, in the presence of an evaluation operator, whether or not a variable is free in an expression may depend on the values of the expression's components. As a consequence, the substitution of an expression for the free occurrences of a variable in another expression depends on the semantics (as well as the syntax) of the expressions involved and must be integrated into the proof system of the logic. Hence a logic with quotation and evaluation requires a semantics-dependent form of substitution. This is a major departure from traditional logic.

See [25] for a more detailed presentation of the replete approach including discussion of some other problems that come with it.<sup>1</sup>

There are several implementations of the replete approach in programming languages. The most well-known example is the Lisp programming language with its quote and eval operators. Other examples are Agda [49, 50], Archon [60], F# [65], MetaML [61], MetaOCaml [58], reFLect [35], and Template Haskell [59].

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<sup>1</sup>The *replete approach* is called the *global approach* in [25].

Implementations of the replete approach are much rarer in logics. One example is a logic called Chiron [21, 22] which is a derivative of von-Neumann-Bernays-Gödel (NBG) set theory. It admits undefined expressions, has a rich type system, and contains the machinery of a replete syntax framework. As far as we know, there is no implementation of the replete approach in simple type theory. See [30, 45] for research moving in this direction. Such an implementation would require significant changes to the logic:

1. A syntax language that represents the set of expressions of the logic must be defined in the logic.
2. The syntax and semantics of the logic must be modified to admit quotation and evaluation operators.
3. The proof system of the logic must be extended to include the means to reason about quotation, evaluation, and substitution.

Moreover, these changes must provide solutions to the Evaluation and Variable Problems.

The purpose of this paper is to demonstrate how the replete approach can be implemented in Church's type theory [12], a version of simple type theory with lambda-notation introduced by Alonzo Church in 1940. We start with  $\mathcal{Q}_0$ , an especially elegant version of Church's type theory formulated by Peter B. Andrews and meticulously described and analyzed in [2]. Since evaluation unavoidably leads to undefined expressions, we modify  $\mathcal{Q}_0$  so that it formalizes the traditional approach to undefinedness [19]. This version of  $\mathcal{Q}_0$  with undefined expressions called  $\mathcal{Q}_0^u$  is presented in [23]. ( $\mathcal{Q}_0^u$  is a simplified version of LUTINS [16, 17, 18], the logic of the the IMPS theorem proving system [26, 27].) And, finally, we modify  $\mathcal{Q}_0^u$  so that it implements the replete approach. This version of  $\mathcal{Q}_0$  with undefined expressions, quotation, and evaluation called  $\mathcal{Q}_0^{uqe}$  is presented in this paper.

$\mathcal{Q}_0^{uqe}$  consists of three principal components: a syntax, a semantics, and a proof system. The syntax and semantics of  $\mathcal{Q}_0^{uqe}$  are relatively straightforward extensions of the syntax and semantics of  $\mathcal{Q}_0^u$ . However, the proof system of  $\mathcal{Q}_0^{uqe}$  is significantly more complicated than the proof system of  $\mathcal{Q}_0^u$ . This is because the Variable Problem discussed above necessitates that the proof system employ a semantics-dependent substitution mechanism. The proof system of  $\mathcal{Q}_0^{uqe}$  can be used to effectively reason about quotations and evaluations, but unlike the proof systems of  $\mathcal{Q}_0$  and  $\mathcal{Q}_0^u$  it is not complete. However, we do show that it is complete for formulas that do not contain evaluations.

The paper is organized as follows. The syntax of  $\mathcal{Q}_0^{uqe}$  is defined in section 2. A Henkin-style general models semantics for  $\mathcal{Q}_0^{uqe}$  is presented in section 3. Section 4 introduces several important defined logical constants and abbreviations. Section 5 shows that  $\mathcal{Q}_0^{uqe}$  embodies the structure of a replete syntax framework. Section 6 finishes the specification of the logical constants of  $\mathcal{Q}_0^{uqe}$  and defines the notion of a normal general model for  $\mathcal{Q}_0^{uqe}$ . The substitution mechanism for  $\mathcal{Q}_0^{uqe}$  is presented in section 7. Section 8 defines  $\mathcal{P}^{uqe}$ , the proof system of

$\mathcal{Q}_0^{\text{uqe}}$ .  $\mathcal{P}^{\text{uqe}}$  is proved in section 9 to be sound with respect to the semantics of  $\mathcal{Q}_0^{\text{uqe}}$ . Several metatheorems of  $\mathcal{P}^{\text{uqe}}$  are proved in section 10.  $\mathcal{P}^{\text{uqe}}$  is proved in section 11 to be complete with respect to the semantics of  $\mathcal{Q}_0^{\text{uqe}}$  for evaluation-free formulas. Some applications of  $\mathcal{Q}_0^{\text{uqe}}$  are illustrated in section 12. And the paper ends with some final remarks in section 13 including a brief discussion on related and future work.

The great majority of the definitions for  $\mathcal{Q}_0^{\text{uqe}}$  are derived from those for  $\mathcal{Q}_0$  given in [2]. In fact, many  $\mathcal{Q}_0^{\text{uqe}}$  definitions are exactly the same as the  $\mathcal{Q}_0$  definitions. Of these, we repeat only the most important and least obvious definitions for  $\mathcal{Q}_0$ ; for the others the reader is referred to [2].

## 2 Syntax

The syntax of  $\mathcal{Q}_0^{\text{uqe}}$  includes the syntax of  $\mathcal{Q}_0^{\text{u}}$  plus machinery for reasoning about the syntax of expressions (i.e., wffs in Andrews' terminology) based on quotation and evaluation.

### 2.1 Symbols

A *type symbol* of  $\mathcal{Q}_0^{\text{uqe}}$  is defined inductively as follows:

1.  $\iota$  is a type symbol.
2.  $\circ$  is a type symbol.
3.  $\epsilon$  is a type symbol.
4. If  $\alpha$  and  $\beta$  are type symbols, then  $(\alpha\beta)$  is a type symbol.
5. If  $\alpha$  and  $\beta$  are type symbols, then  $\langle\alpha\beta\rangle$  is a type symbol.

Let  $\mathcal{T}$  denote the set of type symbols.  $\alpha, \beta, \gamma, \dots$  are syntactic variables ranging over type symbols. When there is no loss of meaning, matching pairs of parentheses in type symbols may be omitted. We assume that type combination of the form  $(\alpha\beta)$  associates to the left so that a type of the form  $((\alpha\beta)\gamma)$  may be written as  $\alpha\beta\gamma$ .

The *primitive symbols* of  $\mathcal{Q}_0^{\text{uqe}}$  are the following:

1. *Improper symbols*:  $[, ], \lambda, \mathbf{c}, \mathbf{q}, \mathbf{e}$ .
2. A denumerable set of *variables* of type  $\alpha$  for each  $\alpha \in \mathcal{T}$ :  $f_\alpha, g_\alpha, h_\alpha, x_\alpha, y_\alpha, z_\alpha, f_\alpha^1, g_\alpha^1, h_\alpha^1, x_\alpha^1, y_\alpha^1, z_\alpha^1, \dots$ .
3. *Logical constants*: see Table 1.
4. An unspecified set of *nonlogical constants* of various types.

The types of variables and constants are indicated by their subscripts.  $\mathbf{f}_\alpha, \mathbf{g}_\alpha, \mathbf{h}_\alpha, \mathbf{x}_\alpha, \mathbf{y}_\alpha, \mathbf{z}_\alpha, \dots$  are syntactic variables ranging over variables of type  $\alpha$ .

$Q_{((o\alpha)\alpha)}$	for all $\alpha \in \mathcal{T}$
$\iota_{(\alpha(o\alpha))}$	for all $\alpha \in \mathcal{T}$ with $\alpha \neq o$
$\text{pair}_{((\langle\alpha\beta\rangle\beta)\alpha)}$	for all $\alpha, \beta \in \mathcal{T}$
$\text{var}_{(o\epsilon)}$	
$\text{con}_{(o\epsilon)}$	
$\text{app}_{((\epsilon\epsilon)\epsilon)}$	
$\text{abs}_{((\epsilon\epsilon)\epsilon)}$	
$\text{cond}_{(((\epsilon\epsilon)\epsilon)\epsilon)}$	
$\text{quot}_{(\epsilon\epsilon)}$	
$\text{eval}_{((\epsilon\epsilon)\epsilon)}$	
$\text{eval-free}_{(o\epsilon)}$	
$\text{not-free-in}_{((o\epsilon)\epsilon)}$	
$\text{cleanse}_{(\epsilon\epsilon)}$	
$\text{sub}_{(((\epsilon\epsilon)\epsilon)\epsilon)}$	
$\text{wff}_{(o\epsilon)}^\alpha$	for all $\alpha \in \mathcal{T}$

Table 1: Logical Constants

**Note 1 (Iota Constants)** Only  $\iota_{\alpha(o\alpha)}$  is a primitive logical constant in  $\mathcal{Q}_0$ ; each other  $\iota_{\alpha(o\alpha)}$  is a nonprimitive logical constant in  $\mathcal{Q}_0$  defined according to an inductive scheme presented by Church in [12] (see [2, pp. 233–4]). We will see in the next section that the iota constants have a different semantics in  $\mathcal{Q}_0^{\text{uqe}}$  than in  $\mathcal{Q}_0$ . As a result, it is not possible to define the iota constants in  $\mathcal{Q}_0^{\text{uqe}}$  as they are defined in  $\mathcal{Q}_0$ , and thus they must be primitive in  $\mathcal{Q}_0^{\text{uqe}}$ . Notice that  $\iota_{o(o\alpha)}$  is not a primitive logical constant of  $\mathcal{Q}_0^{\text{uqe}}$ . It has been left out because it serves no useful purpose. It can be defined as a nonprimitive logical constant as in [2, p. 233] if desired.

## 2.2 Wffs

Following Andrews, we will call the expressions of  $\mathcal{Q}_0^{\text{uqe}}$  *well-formed formulas (wffs)*. We are now ready to define a *wff of type  $\alpha$*  ( $\text{wff}_\alpha$ ) of  $\mathcal{Q}_0^{\text{uqe}}$ .  $\mathbf{A}_\alpha, \mathbf{B}_\alpha, \mathbf{C}_\alpha, \dots$  are syntactic variables ranging over wffs of type  $\alpha$ . A  $\text{wff}_\alpha$  is defined inductively as follows:

1. A variable of type  $\alpha$  is a  $\text{wff}_\alpha$ .
2. A primitive constant of type  $\alpha$  is a  $\text{wff}_\alpha$ .
3.  $[\mathbf{A}_{\alpha\beta}\mathbf{B}_\beta]$  is a  $\text{wff}_\alpha$ .
4.  $[\lambda\mathbf{x}_\beta\mathbf{A}_\alpha]$  is a  $\text{wff}_{\alpha\beta}$ .
5.  $[\mathbf{c}\mathbf{A}_o\mathbf{B}_\alpha\mathbf{C}_\alpha]$  is a  $\text{wff}_\alpha$ .
6.  $[\mathbf{q}\mathbf{A}_\alpha]$  is a  $\text{wff}_\epsilon$ .



7.  $[\mathbf{eA}_\epsilon \mathbf{x}_\alpha]$  is a wff $_\alpha$ .

A wff of the form  $[\mathbf{A}_{\alpha\beta} \mathbf{B}_\beta]$ ,  $[\lambda \mathbf{x}_\beta \mathbf{A}_\alpha]$ ,  $[\mathbf{cA}_o \mathbf{B}_\alpha \mathbf{C}_\alpha]$ ,  $[\mathbf{qA}_\alpha]$ , or  $[\mathbf{eA}_\epsilon \mathbf{x}_\alpha]$  is called a *function application*, a *function abstraction*, a *conditional*, a *quotation*, or an *evaluation*, respectively. A *formula* is a wff $_o$ .  $\mathbf{A}_\alpha$  is *evaluation-free* if each occurrence of an evaluation in  $\mathbf{A}_\alpha$  is within a quotation. When there is no loss of meaning, matching pairs of square brackets in wffs may be omitted. We assume that wff combination of the form  $[\mathbf{A}_{\alpha\beta} \mathbf{B}_\beta]$  associates to the left so that a wff  $[[\mathbf{C}_{\gamma\beta\alpha} \mathbf{A}_\alpha] \mathbf{B}_\beta]$  may be written as  $\mathbf{C}_{\gamma\beta\alpha} \mathbf{A}_\alpha \mathbf{B}_\beta$ .

The *size* of  $\mathbf{A}_\alpha$  is the number of variables and primitive constants occurring in  $\mathbf{A}_\alpha$ . The *complexity* of  $\mathbf{A}_\alpha$  is the ordered pair  $(m, n)$  of natural numbers such that  $m$  is the number of evaluations occurring in  $\mathbf{A}_\alpha$  that are not within a quotation and  $n$  is the size of  $\mathbf{A}_\alpha$ . Complexity pairs are ordered lexicographically. The complexity of an evaluation-free wff is a pair  $(0, n)$  where  $n$  is the size of the wff.

**Note 2 (Type  $\epsilon$ )** The type  $\epsilon$  denotes an inductively defined set  $\mathcal{D}_\epsilon$  of values called *constructions* that represent the syntactic structures of wffs. The constants  $\mathbf{app}_{\epsilon\epsilon\epsilon}$ ,  $\mathbf{abs}_{\epsilon\epsilon\epsilon}$ ,  $\mathbf{cond}_{\epsilon\epsilon\epsilon\epsilon}$ ,  $\mathbf{quot}_{\epsilon\epsilon}$ , and  $\mathbf{eval}_{\epsilon\epsilon\epsilon}$  are used to build wffs that denote constructions representing function applications, function abstractions, conditionals, quotations, and evaluations, respectively. An alternate approach would be to have a type  $\epsilon_\alpha$  of constructions that represent the syntactic structures of wffs $_\alpha$  for each  $\alpha \in \mathcal{T}$ .

**Note 3 (Type  $(\alpha\beta)$ )** A type  $(\alpha\beta)$  denotes a set of partial and total functions from values of  $\alpha$  to values of type  $\beta$ .  $\beta \rightarrow \alpha$  is an alternate notation for  $(\alpha\beta)$ .

**Note 4 (Type  $\langle\alpha\beta\rangle$ )** A type  $\langle\alpha\beta\rangle$  denotes the set of ordered pairs  $\langle a, b \rangle$  where  $a$  is a value of type  $\alpha$  and  $b$  is a value of type  $\beta$ .  $\alpha \times \beta$  is an alternate notation for  $\langle\alpha\beta\rangle$ . The constant  $\mathbf{pair}_{\langle\alpha\beta\rangle\beta\alpha}$  is used to construct ordered pairs of type  $\langle\alpha\beta\rangle$ .

**Note 5 (Conditionals)** We will see that  $[\mathbf{cA}_o \mathbf{B}_\alpha \mathbf{C}_\alpha]$  is a conditional that is not strict with respect to undefinedness. For instance, if  $\mathbf{A}_o$  is true, then  $[\mathbf{cA}_o \mathbf{B}_\alpha \mathbf{C}_\alpha]$  denotes the value of  $\mathbf{B}_\alpha$  even when  $\mathbf{C}_\alpha$  is undefined. We construct conditionals using a primitive wff constructor instead of using a primitive or defined constant since constants always denote functions that are effectively strict with respect to undefinedness.

**Note 6 (Evaluation Syntax)** The sole purpose of the variable  $\mathbf{x}_\alpha$  in an evaluation  $[\mathbf{eA}_\epsilon \mathbf{x}_\alpha]$  is to designate the type of the evaluation. We will see in the next section that this evaluation is defined (true if  $\alpha = o$ ) only if  $\mathbf{A}_\epsilon$  denotes a construction representing a wff $_\alpha$ . Hence, if  $\mathbf{A}_\epsilon$  does denote a construction representing a wff $_\alpha$ ,  $[\mathbf{eA}_\epsilon \mathbf{x}_\beta]$  is undefined (false if  $\alpha = o$ ) for all  $\beta \in \mathcal{T}$  with  $\beta \neq \alpha$ .

Kind	Syntax	Syntactic Representation
Variable	$\mathbf{x}_\alpha$	$[\mathbf{q}\mathbf{x}_\alpha]$
Primitive constant	$\mathbf{c}_\alpha$	$[\mathbf{q}\mathbf{c}_\alpha]$
Function application	$[\mathbf{A}_{\alpha\beta}\mathbf{B}_\beta]$	$[\mathbf{app}_{\epsilon\epsilon\epsilon}\mathcal{E}(\mathbf{A}_{\alpha\beta})\mathcal{E}(\mathbf{B}_\beta)]$
Function abstraction	$[\lambda\mathbf{x}_\beta\mathbf{A}_\alpha]$	$[\mathbf{abs}_{\epsilon\epsilon\epsilon}\mathcal{E}(\mathbf{x}_\beta)\mathcal{E}(\mathbf{A}_\alpha)]$
Conditional	$[\mathbf{c}\mathbf{A}_o\mathbf{B}_\alpha\mathbf{C}_\alpha]$	$[\mathbf{cond}_{\epsilon\epsilon\epsilon\epsilon}\mathcal{E}(\mathbf{A}_o)\mathcal{E}(\mathbf{B}_\alpha)\mathcal{E}(\mathbf{C}_\alpha)]$
Quotation	$[\mathbf{q}\mathbf{A}_\alpha]$	$[\mathbf{quot}_{\epsilon\epsilon}\mathcal{E}(\mathbf{A}_\alpha)]$
Evaluation	$[\mathbf{e}\mathbf{A}_\epsilon\mathbf{x}_\alpha]$	$[\mathbf{eval}_{\epsilon\epsilon\epsilon}\mathcal{E}(\mathbf{A}_\epsilon)\mathcal{E}(\mathbf{x}_\alpha)]$

Table 2: Seven Kinds of Wffs

### 3 Semantics

The semantics of  $\mathcal{Q}_0^{\text{uqe}}$  is obtained by making three principal changes to the semantics of  $\mathcal{Q}_0^{\text{u}}$ : (1) The semantics of the type  $\epsilon$  is defined to be a domain  $D_\epsilon$  of values such that, for each wff  $\mathbf{A}_\alpha$  of  $\mathcal{Q}_0^{\text{uqe}}$ , there is a unique member of  $D_\epsilon$  that represents the syntactic structure of  $\mathbf{A}_\alpha$ . (2) The semantics of the type constructor  $\langle\alpha\beta\rangle$  is defined to be a domain of ordered pairs. (3) The valuation function for wffs is extended to include conditionals, quotations, and evaluations in its domain.

#### 3.1 Frames

Let  $\mathcal{E}$  be the function from the set of wffs to the set of wffs $_\epsilon$  defined inductively as follows:

1.  $\mathcal{E}(\mathbf{x}_\alpha) = [\mathbf{q}\mathbf{x}_\alpha]$ .
2.  $\mathcal{E}(\mathbf{c}_\alpha) = [\mathbf{q}\mathbf{c}_\alpha]$  where  $\mathbf{c}_\alpha$  is a primitive constant.
3.  $\mathcal{E}([\mathbf{A}_{\alpha\beta}\mathbf{B}_\beta]) = [\mathbf{app}_{\epsilon\epsilon\epsilon}\mathcal{E}(\mathbf{A}_{\alpha\beta})\mathcal{E}(\mathbf{B}_\beta)]$ .
4.  $\mathcal{E}([\lambda\mathbf{x}_\beta\mathbf{A}_\alpha]) = [\mathbf{abs}_{\epsilon\epsilon\epsilon}\mathcal{E}(\mathbf{x}_\beta)\mathcal{E}(\mathbf{A}_\alpha)]$ .
5.  $\mathcal{E}([\mathbf{c}\mathbf{A}_o\mathbf{B}_\alpha\mathbf{C}_\alpha]) = [\mathbf{cond}_{\epsilon\epsilon\epsilon\epsilon}\mathcal{E}(\mathbf{A}_o)\mathcal{E}(\mathbf{B}_\alpha)\mathcal{E}(\mathbf{C}_\alpha)]$ .
6.  $\mathcal{E}([\mathbf{q}\mathbf{A}_\alpha]) = [\mathbf{quot}_{\epsilon\epsilon}\mathcal{E}(\mathbf{A}_\alpha)]$ .
7.  $\mathcal{E}([\mathbf{e}\mathbf{A}_\epsilon\mathbf{x}_\alpha]) = [\mathbf{eval}_{\epsilon\epsilon\epsilon}\mathcal{E}(\mathbf{A}_\epsilon)\mathcal{E}(\mathbf{x}_\alpha)]$ .

$\mathcal{E}$  is obviously an injective, total function whose range is a proper subset of the set of wffs $_\epsilon$ . The wff $_\epsilon$   $\mathcal{E}(\mathbf{A}_\alpha)$  represents the syntactic structure of the wff  $\mathbf{A}_\alpha$ . The seven kinds of wffs and their syntactic representations are given in Table 2.

A *frame* of  $\mathcal{Q}_0^{\text{uqe}}$  is a collection  $\{\mathcal{D}_\alpha \mid \alpha \in \mathcal{T}\}$  of nonempty domains such that:

1.  $\mathcal{D}_o = \{\mathbf{T}, \mathbf{F}\}$ .

2.  $\{\mathcal{E}(\mathbf{A}_\alpha) \mid \mathbf{A}_\alpha \text{ is a wff}\} \subseteq \mathcal{D}_\epsilon$ .
3. For  $\alpha, \beta \in \mathcal{T}$ ,  $\mathcal{D}_{\langle\alpha\beta\rangle}$  is some set of *total* functions from  $\mathcal{D}_\beta$  to  $\mathcal{D}_\alpha$  if  $\alpha = o$  and is some set of *partial and total* functions from  $\mathcal{D}_\beta$  to  $\mathcal{D}_\alpha$  if  $\alpha \neq o$ .
4. For  $\alpha, \beta \in \mathcal{T}$ ,  $\mathcal{D}_{\langle\alpha\beta\rangle}$  is the set of all ordered pairs  $\langle a, b \rangle$  such that  $a \in \mathcal{D}_\alpha$  and  $b \in \mathcal{D}_\beta$ .

$\mathcal{D}_i$  is the *domain of individuals*,  $\mathcal{D}_o$  is the *domain of truth values*,  $\mathcal{D}_\epsilon$  is the *domain of constructions*, and, for  $\alpha, \beta \in \mathcal{T}$ ,  $\mathcal{D}_{\langle\alpha\beta\rangle}$  is a *function domain* and  $\mathcal{D}_{\langle\alpha\beta\rangle}$  is a *ordered pair domain*. For all  $\alpha \in \mathcal{T}$ , the *identity relation* on  $\mathcal{D}_\alpha$  is the total function  $q \in \mathcal{D}_{o\alpha\alpha}$  such that, for all  $x, y \in \mathcal{D}_\alpha$ ,  $q(x)(y) = \top$  iff  $x = y$ . For all  $\alpha \in \mathcal{T}$  with  $\alpha \neq o$ , the *unique member selector* on  $\mathcal{D}_\alpha$  is the partial function  $f \in \mathcal{D}_{\alpha(o\alpha)}$  such that, for all  $s \in \mathcal{D}_{o\alpha}$ , if the predicate  $s$  represents a singleton  $\{x\} \subseteq \mathcal{D}_\alpha$ , then  $f(s) = x$ , and otherwise  $f(s)$  is undefined. For all  $\alpha, \beta \in \mathcal{T}$ , the *pairing function* on  $\mathcal{D}_\alpha$  and  $\mathcal{D}_\beta$  is the total function  $f \in \mathcal{D}_{\langle\alpha\beta\rangle\beta\alpha}$  such that, for all  $a \in \mathcal{D}_\alpha$  and  $b \in \mathcal{D}_\beta$ ,  $f(a)(b) = \langle a, b \rangle$ , the ordered pair of  $a$  and  $b$ .

**Note 7 (Function Domains)** In a  $\mathcal{Q}_0$  frame a function domain  $\mathcal{D}_{\alpha\beta}$  contains only total functions, while in a  $\mathcal{Q}_0^{\text{uqe}}$  (and  $\mathcal{Q}_0^{\text{u}}$ ) frame a function domain  $\mathcal{D}_{o\beta}$  contains only total functions but a function domain  $\mathcal{D}_{\alpha\beta}$  with  $\alpha \neq o$  contains partial functions as well as total functions.

## 3.2 Interpretations

An *interpretation*  $\langle \{\mathcal{D}_\alpha \mid \alpha \in \mathcal{T}\}, \mathcal{J} \rangle$  of  $\mathcal{Q}_0^{\text{uqe}}$  consists of a frame and an interpretation function  $\mathcal{J}$  that maps each primitive constant of  $\mathcal{Q}_0^{\text{uqe}}$  of type  $\alpha$  to an element of  $\mathcal{D}_\alpha$  such that:

1.  $\mathcal{J}(\mathbf{Q}_{o\alpha\alpha})$  is the identity relation on  $\mathcal{D}_\alpha$  for all  $\alpha \in \mathcal{T}$ .
2.  $\mathcal{J}(\iota_{\alpha(o\alpha)})$  is the unique member selector on  $\mathcal{D}_\alpha$  for all  $\alpha \in \mathcal{T}$  with  $\alpha \neq o$ .
3.  $\mathcal{J}(\text{pair}_{\langle\alpha\beta\rangle\beta\alpha})$  is the pairing function on  $\mathcal{D}_\alpha$  and  $\mathcal{D}_\beta$  for all  $\alpha, \beta \in \mathcal{T}$ .

The other 12 logical constants involving the type  $\epsilon$  will be specified later via axioms in section 6.1.

**Note 8 (Definite Description Operators)** The  $\iota_{\alpha(o\alpha)}$  in  $\mathcal{Q}_0$  are *description operators*: if  $\mathbf{A}_{o\alpha}$  denotes a singleton, then the value of  $\iota_{\alpha(o\alpha)}\mathbf{A}_{o\alpha}$  is the unique member of the singleton, and otherwise the value of  $\iota_{\alpha(o\alpha)}\mathbf{A}_{o\alpha}$  is *unspecified*. In contrast, the  $\iota_{\alpha(o\alpha)}$  in  $\mathcal{Q}_0^{\text{uqe}}$  (and  $\mathcal{Q}_0^{\text{u}}$ ) are *definite description operators*: if  $\mathbf{A}_{o\alpha}$  denotes a singleton, then the value of  $\iota_{\alpha(o\alpha)}\mathbf{A}_{o\alpha}$  is the unique member of the singleton, and otherwise the value of  $\iota_{\alpha(o\alpha)}\mathbf{A}_{o\alpha}$  is *undefined*.

An *assignment* into a frame  $\{\mathcal{D}_\alpha \mid \alpha \in \mathcal{T}\}$  is a function  $\varphi$  whose domain is the set of variables of  $\mathcal{Q}_0^{\text{uqe}}$  such that, for each variable  $\mathbf{x}_\alpha$ ,  $\varphi(\mathbf{x}_\alpha) \in \mathcal{D}_\alpha$ . Given an assignment  $\varphi$ , a variable  $\mathbf{x}_\alpha$ , and  $d \in \mathcal{D}_\alpha$ , let  $\varphi[\mathbf{x}_\alpha \mapsto d]$  be the assignment  $\psi$  such that  $\psi(\mathbf{x}_\alpha) = d$  and  $\psi(\mathbf{y}_\beta) = \varphi(\mathbf{y}_\beta)$  for all variables  $\mathbf{y}_\beta \neq \mathbf{x}_\alpha$ . Given an interpretation  $\mathcal{M} = \langle \{\mathcal{D}_\alpha \mid \alpha \in \mathcal{T}\}, \mathcal{J} \rangle$ ,  $\text{assign}(\mathcal{M})$  is the set of assignments into the frame of  $\mathcal{M}$ .

### 3.3 General and Evaluation-Free Models

An interpretation  $\mathcal{M} = \langle \{\mathcal{D}_\alpha \mid \alpha \in \mathcal{T}\}, \mathcal{J} \rangle$  is a *general model* for  $\mathcal{Q}_0^{\text{uqe}}$  if there is a binary valuation function  $\mathcal{V}^{\mathcal{M}}$  such that, for each assignment  $\varphi \in \text{assign}(\mathcal{M})$  and wff  $\mathbf{D}_\delta$ , either  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta) \in \mathcal{D}_\delta$  or  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta)$  is undefined and the following conditions are satisfied for all assignments  $\varphi \in \text{assign}(\mathcal{M})$  and all wffs  $\mathbf{D}_\delta$ :

1. Let  $\mathbf{D}_\delta$  be a variable of  $\mathcal{Q}_0^{\text{uqe}}$ . Then  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta) = \varphi(\mathbf{D}_\delta)$ .
2. Let  $\mathbf{D}_\delta$  be a primitive constant of  $\mathcal{Q}_0^{\text{uqe}}$ . Then  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta) = \mathcal{J}(\mathbf{D}_\delta)$ .
3. Let  $\mathbf{D}_\delta$  be  $[\mathbf{A}_{\alpha\beta}\mathbf{B}_\beta]$ . If  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_{\alpha\beta})$  is defined,  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_\beta)$  is defined, and the function  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_{\alpha\beta})$  is defined at the argument  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_\beta)$ , then

$$\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta) = \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_{\alpha\beta})(\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_\beta)),$$

the value of the function  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_{\alpha\beta})$  at the argument  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_\beta)$ . Otherwise,  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta) = \mathbf{F}$  if  $\alpha = o$  and  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta)$  is undefined if  $\alpha \neq o$ .

4. Let  $\mathbf{D}_\delta$  be  $[\lambda\mathbf{x}_\beta\mathbf{B}_\alpha]$ . Then  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta)$  is the (partial or total) function  $f \in \mathcal{D}_{\alpha\beta}$  such that, for each  $d \in \mathcal{D}_\beta$ ,  $f(d) = \mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto d]}^{\mathcal{M}}(\mathbf{B}_\alpha)$  if  $\mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto d]}^{\mathcal{M}}(\mathbf{B}_\alpha)$  is defined and  $f(d)$  is undefined if  $\mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto d]}^{\mathcal{M}}(\mathbf{B}_\alpha)$  is undefined.
5. Let  $\mathbf{D}_\delta$  be  $[\mathbf{cA}_o\mathbf{B}_\alpha\mathbf{C}_\alpha]$ . If  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_o) = \mathbf{T}$  and  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_\alpha)$  is defined, then  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta) = \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_\alpha)$ . If  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_o) = \mathbf{T}$  and  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_\alpha)$  is undefined, then  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta)$  is undefined. If  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_o) = \mathbf{F}$  and  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{C}_\alpha)$  is defined, then  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta) = \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{C}_\alpha)$ . If  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_o) = \mathbf{F}$  and  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{C}_\alpha)$  is undefined, then  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta)$  is undefined.
6. Let  $\mathbf{D}_\delta$  be  $[\mathbf{qA}_\alpha]$ . Then  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta) = \mathcal{E}(\mathbf{A}_\alpha)$ .
7. Let  $\mathbf{D}_\delta$  be  $[\mathbf{eA}_\epsilon\mathbf{x}_\alpha]$ . If  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\epsilon)$  is defined,  $\mathcal{E}^{-1}(\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\epsilon))$  is an evaluation-free wff $_\alpha$ , and  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathcal{E}^{-1}(\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\epsilon)))$  is defined, then

$$\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta) = \mathcal{V}_\varphi^{\mathcal{M}}(\mathcal{E}^{-1}(\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\epsilon))).$$

Otherwise,  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta) = \mathbf{F}$  if  $\alpha = o$  and  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta)$  is undefined if  $\alpha \neq o$ .

**Proposition 3.3.1** *General models for  $\mathcal{Q}_0^{\text{uqe}}$  exist.*

**Proof** It is easy to construct an interpretation  $\mathcal{M} = \langle \{\mathcal{D}_\alpha \mid \alpha \in \mathcal{T}\}, \mathcal{J} \rangle$  that is a general model for  $\mathcal{Q}_0^{\text{uqe}}$  for which  $\mathcal{D}_{\alpha\beta}$  is the set of all *total* functions from  $\mathcal{D}_\beta$  to  $\mathcal{D}_\alpha$  if  $\alpha = o$  and is the set of all *partial and total* functions from  $\mathcal{D}_\beta$  to  $\mathcal{D}_\alpha$  if  $\alpha \neq o$  for all  $\alpha, \beta \in \mathcal{T}$ .  $\square$

An interpretation  $\mathcal{M} = \langle \{\mathcal{D}_\alpha \mid \alpha \in \mathcal{T}\}, \mathcal{J} \rangle$  is an *evaluation-free model* for  $\mathcal{Q}_0^{\text{uqe}}$  if there is a binary valuation function  $\mathcal{V}^{\mathcal{M}}$  such that, for each assignment

$\varphi \in \text{assign}(\mathcal{M})$  and evaluation-free wff  $\mathbf{D}_\delta$ , either  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta) \in \mathcal{D}_\delta$  or  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta)$  is undefined and conditions 1–6 above are satisfied for all assignments  $\varphi \in \text{assign}(\mathcal{M})$  and all evaluation-free wffs  $\mathbf{D}_\delta$ . A general model is also an evaluation-free model.

**Note 9 (Valuation Function)** In  $\mathcal{Q}_0$ , if  $\mathcal{M}$  is a general model, then  $\mathcal{V}^{\mathcal{M}}$  is total and the value of  $\mathcal{V}^{\mathcal{M}}$  on a function abstraction is always a total function. In  $\mathcal{Q}_0^{\text{uqe}}$ , if  $\mathcal{M}$  is a general model, then  $\mathcal{V}^{\mathcal{M}}$  is partial and the value of  $\mathcal{V}^{\mathcal{M}}$  on a function abstraction can be either a partial or a total function.

**Proposition 3.3.2** *Let  $\mathcal{M}$  be a general model for  $\mathcal{Q}_0^{\text{uqe}}$ . Then  $\mathcal{V}^{\mathcal{M}}$  is defined on all variables, primitive constants, function applications of type  $o$ , function abstractions, conditionals of type  $o$ , quotations, and evaluations of type  $o$  and is defined on only a proper subset of function applications of type  $\alpha \neq o$ , a proper subset of conditionals of type  $\alpha \neq o$ , and a proper subset of evaluations of type  $\alpha \neq o$ .*

**Note 10 (Traditional Approach)**  $\mathcal{Q}_0^{\text{uqe}}$  satisfies the three principles of the traditional approach to undefinedness stated in [19]. Like other traditional logics,  $\mathcal{Q}_0$  only satisfies the first principle.

**Note 11 (Theories of Quotation)** The semantics of the quotation operator  $q$  is based on the *disquotational theory of quotation* [9]. According to this theory, a quotation of an expression  $e$  is an expression that denotes  $e$  itself. In our definition of a syntax framework,  $[q\mathbf{A}_\alpha]$  denotes a value that represents  $\mathbf{A}_\alpha$  as a syntactic entity. Andrew Polonsky presents in [54] a set of axioms for quotation operators of this kind. There are several other theories of quotation that have been proposed [9].

**Note 12 (Theories of Truth)**  $[e\mathbf{A}_\epsilon\mathbf{x}_o]$  asserts the truth of the formula represented by  $\mathbf{A}_\epsilon$ . Thus the evaluation operator  $e$  is a *truth predicate* [32]. A truth predicate is the face of a *theory of truth*: the properties of a truth predicate characterize a theory of truth [43]. What truth is and how it can be formalized is a fundamental research area of logic, and avoiding inconsistencies derived from the liar paradox and similar statements is one of the major research issues in the area (see [37]).

**Note 13 (Evaluation Semantics)** An evaluation of type  $\alpha$  is undefined (false if  $\alpha = o$ ) whenever its (first) argument represents a non-evaluation-free wff $_\alpha$ . This idea avoids the Evaluation Problem discussed in section 1. The origin of this idea is found in Tarski’s famous paper on the concept of truth [62, 63, 64, Theorem III]. See [36] for a different approach for overcoming the Evaluation Problem in which the argument of an evaluation is restricted to wffs that only contain positive occurrences of evaluations.

$\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha) \simeq \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_\alpha)$  means either  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)$  and  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_\alpha)$  are both defined and equal or  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)$  and  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_\alpha)$  are both undefined. Given a set  $X$  of

variables,  $\mathbf{A}_\alpha$  is *independent of  $X$  in  $\mathcal{M}$*  if  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha) \simeq \mathcal{V}_{\varphi'}^{\mathcal{M}}(\mathbf{A}_\alpha)$  for all  $\varphi, \varphi' \in \text{assign}(\mathcal{M})$  such that  $\varphi(\mathbf{x}_\alpha) = \varphi'(\mathbf{x}_\alpha)$  whenever  $\mathbf{x}_\alpha \notin X$ .  $\mathbf{A}_\alpha$  is *semantically closed* if  $\mathbf{A}_\alpha$  is independent of  $X$  in every general model for  $\mathcal{Q}_0^{\text{uqe}}$  where  $X$  is the set of all variables. A *sentence* is a semantically closed formula.  $\mathbf{A}_\alpha$  is *invariable* if  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)$  is the same value or undefined for every general model  $\mathcal{M}$  for  $\mathcal{Q}_0^{\text{uqe}}$  and every  $\varphi \in \text{assign}(\mathcal{M})$ . If  $\mathbf{A}_\alpha$  is invariable,  $\mathbf{A}_\alpha$  is said to *denote* the value  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)$  when  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)$  is defined and to be *undefined* otherwise.

**Proposition 3.3.3** *A wff that contains variables only within a quotation or the second argument of an evaluation is semantically closed.*

**Proposition 3.3.4** *Quotations and tautologous formulas are invariable.*

Let  $\mathcal{H}$  be a set of wffs<sub>o</sub> and  $\mathcal{M}$  be a general model for  $\mathcal{Q}_0^{\text{uqe}}$ .  $\mathbf{A}_o$  is *valid* in  $\mathcal{M}$ , written  $\mathcal{M} \models \mathbf{A}_o$ , if  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_o) = \top$  for all assignments  $\varphi \in \text{assign}(\mathcal{M})$ .  $\mathcal{M}$  is a *general model for  $\mathcal{H}$* , written  $\mathcal{M} \models \mathcal{H}$ , if  $\mathcal{M} \models \mathbf{B}_o$  for all  $\mathbf{B}_o \in \mathcal{H}$ . We write  $\mathcal{H} \models \mathbf{A}_o$  to mean  $\mathcal{M} \models \mathbf{A}_o$  for every general model  $\mathcal{M}$  for  $\mathcal{H}$ . We write  $\models \mathbf{A}_o$  to mean  $\emptyset \models \mathbf{A}_o$ .

Now let  $\mathbf{A}_o$  be evaluation-free,  $\mathcal{H}$  be a set of evaluation-free wffs<sub>o</sub>, and  $\mathcal{M}$  be an evaluation-free model for  $\mathcal{Q}_0^{\text{uqe}}$ .  $\mathbf{A}_o$  is *valid* in  $\mathcal{M}$ , written  $\mathcal{M} \models \mathbf{A}_o$ , if  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_o) = \top$  for all assignments  $\varphi \in \text{assign}(\mathcal{M})$ .  $\mathcal{M}$  is an *evaluation-free model for  $\mathcal{H}$* , written  $\mathcal{M} \models \mathcal{H}$ , if  $\mathcal{M} \models \mathbf{B}_o$  for all  $\mathbf{B}_o \in \mathcal{H}$ . We write  $\mathcal{H} \models^{\text{ef}} \mathbf{A}_o$  to mean  $\mathcal{M} \models \mathbf{A}_o$  for every evaluation-free model  $\mathcal{M}$  for  $\mathcal{H}$ . We write  $\models^{\text{ef}} \mathbf{A}_o$  to mean  $\emptyset \models^{\text{ef}} \mathbf{A}_o$ .

**Note 14 (Semantically Closed)** Andrews shows in [2] that  $\mathcal{Q}_0$  is undecidable. Hence it is undecidable whether a formula of  $\mathcal{Q}_0$  is valid in all general models for  $\mathcal{Q}_0$ . By similar reasoning, it is undecidable whether a formula of  $\mathcal{Q}_0^{\text{uqe}}$  is valid in all general models for  $\mathcal{Q}_0^{\text{uqe}}$ . This implies that it is undecidable whether a conditional of the form  $\mathbf{cA}_o \mathbf{c}_\alpha \mathbf{x}_\alpha$ , where  $\mathbf{c}_\alpha$  is a primitive constant, is semantically closed. (Primitive constants are semantically closed by Proposition 3.3.3.) Therefore, more generally, it is undecidable whether a given wff is semantically closed. See also Note 18 in section 7.

### 3.4 Standard Models

An interpretation  $\mathcal{M} = \langle \{\mathcal{D}_\alpha \mid \alpha \in \mathcal{T}\}, \mathcal{J} \rangle$  is a *standard model* for  $\mathcal{Q}_0^{\text{uqe}}$  if  $\mathcal{D}_{\alpha\beta}$  is the set of all *total* functions from  $\mathcal{D}_\beta$  to  $\mathcal{D}_\alpha$  if  $\alpha = o$  and is the set of all *partial and total* functions from  $\mathcal{D}_\beta$  to  $\mathcal{D}_\alpha$  if  $\alpha \neq o$  for all  $\alpha, \beta \in \mathcal{T}$ .

**Lemma 3.4.1** *A standard model for  $\mathcal{Q}_0^{\text{uqe}}$  is also a general model for  $\mathcal{Q}_0^{\text{uqe}}$ .*

**Proof** Let  $\mathcal{M}$  be a standard model for  $\mathcal{Q}_0^{\text{uqe}}$ . It is easy to show that  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta)$  is well defined by induction on the complexity of  $\mathbf{D}_\delta$ .  $\square$

By the proof of Proposition 3.3.1, standard models for  $\mathcal{Q}_0^{\text{uqe}}$  exist. A general model for  $\mathcal{Q}_0^{\text{uqe}}$  is a *nonstandard model* for  $\mathcal{Q}_0^{\text{uqe}}$  if it is not a standard model.

## 4 Definitions and Abbreviations

As Andrews does in [2, p. 212], we introduce in Table 3 several defined logical constants and abbreviations. The former includes constants for true and false, the propositional connectives, a canonical undefined wff, the projection functions for pairs, and some predicates for values of type  $\epsilon$ . The latter includes notation for equality, the propositional connectives, universal and existential quantification, defined and undefined wffs, quasi-equality, definite description, conditionals, quotation, and evaluation.

$[\exists_1 \mathbf{x}_\alpha \mathbf{A}_\alpha]$  asserts that there is a unique  $\mathbf{x}_\alpha$  that satisfies  $\mathbf{A}_\alpha$ .

$[\mathbf{I}\mathbf{x}_\alpha \mathbf{A}_\alpha]$  is called a *definite description*. It denotes the unique  $\mathbf{x}_\alpha$  that satisfies  $\mathbf{A}_\alpha$ . If there is no or more than one such  $\mathbf{x}_\alpha$ , it is undefined. Following Bertrand Russell and Church, Andrews denotes this definite description operator as an inverted lower case iota ( $\iota$ ). We represent this operator by an (inverted) capital iota ( $\mathbf{I}$ ).

$[\mathbf{A}_\alpha \downarrow]$  says that  $\mathbf{A}_\alpha$  is defined, and similarly,  $[\mathbf{A}_\alpha \uparrow]$  says that  $\mathbf{A}_\alpha$  is undefined.  $[\mathbf{A}_\alpha \simeq \mathbf{B}_\alpha]$  says that  $\mathbf{A}_\alpha$  and  $\mathbf{B}_\alpha$  are *quasi-equal*, i.e., that  $\mathbf{A}_\alpha$  and  $\mathbf{B}_\alpha$  are either both defined and equal or both undefined. The defined constant  $\perp_\alpha$  is a canonical undefined wff of type  $\alpha$ .

**Note 15 (Definedness Notation)** In  $\mathcal{Q}_0$ ,  $[\mathbf{A}_\alpha \downarrow]$  is always true,  $[\mathbf{A}_\alpha \uparrow]$  is always false,  $[\mathbf{A}_\alpha \simeq \mathbf{B}_\alpha]$  is always equal to  $[\mathbf{A}_\alpha = \mathbf{B}_\alpha]$ , and  $\perp_\alpha$  denotes an unspecified value.

## 5 Syntax Frameworks

In this section we will show that  $\mathcal{Q}_0^{\text{uqe}}$  with a fixed general model and assignment is an instance of a replete syntax framework [28]. We assume that the reader is familiar with the definitions in [28].

Fix a general model  $\mathcal{M} = \langle \{\mathcal{D}_\alpha \mid \alpha \in \mathcal{T}\}, \mathcal{J} \rangle$  for  $\mathcal{Q}_0^{\text{uqe}}$  and an assignment  $\varphi \in \text{assign}(\mathcal{M})$ . Let  $\mathcal{L}$  to be the set of wffs,  $\mathcal{L}_\alpha$  to be the set of wffs $_\alpha$ , and  $\mathcal{D} = \bigcup_\alpha \mathcal{D}_\alpha$ . Choose some value  $\perp \notin \mathcal{D}$ . Define  $\mathcal{W}_\varphi^{\mathcal{M}} : \mathcal{L} \rightarrow \mathcal{D} \cup \{\perp\}$  to be function such that, for all wffs  $\mathbf{D}_\delta$ ,  $\mathcal{W}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta) = \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta)$  if  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta)$  is defined and  $\mathcal{W}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta) = \perp$  otherwise. It is then easy to prove the following three propositions:

**Proposition 5.0.2**  $I = (\mathcal{L}, \mathcal{D} \cup \{\perp\}, \mathcal{W}_\varphi^{\mathcal{M}})$  is an interpreted language.

**Proposition 5.0.3**  $R = (\mathcal{D}_\epsilon \cup \{\perp\}, \mathcal{E})$  is a syntax representation of  $\mathcal{L}$ .

**Proposition 5.0.4**  $(\mathcal{L}_\epsilon, I)$  is a syntax language for  $R$ .

We will now define quotation and evaluation functions. Let  $Q : \mathcal{L} \rightarrow \mathcal{L}_\epsilon$  be the injective, total function that maps each wff  $\mathbf{D}_\delta$  to its quotation  $\ulcorner \mathbf{D}_\delta \urcorner$ . Let  $E : \mathcal{L}_\epsilon \rightarrow \mathcal{L}$  be the partial function that maps each wff $_\epsilon$   $\mathbf{A}_\epsilon$  to  $\llbracket \mathbf{A}_\epsilon \rrbracket_\alpha$  if  $\mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \mathbf{A}_\epsilon \rrbracket_\alpha)$  is defined for some  $\alpha \in \mathcal{T}$  and is undefined otherwise.  $E$  is well defined since  $\mathcal{E}^{-1}(\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\epsilon))$  is a wff of a most one type.

$[\mathbf{A}_\alpha = \mathbf{B}_\alpha]$	stands for	$[\mathbf{Q}_{o\alpha\alpha}\mathbf{A}_\alpha\mathbf{B}_\alpha]$ .
$[\mathbf{A}_o \equiv \mathbf{B}_o]$	stands for	$[\mathbf{Q}_{ooo}\mathbf{A}_o\mathbf{B}_o]$ .
$T_o$	stands for	$[\mathbf{Q}_{ooo} = \mathbf{Q}_{ooo}]$ .
$F_o$	stands for	$[\lambda x_o T_o] = [\lambda x_o x_o]$ .
$[\forall \mathbf{x}_\alpha \mathbf{A}_o]$	stands for	$[\lambda y_\alpha T_o] = [\lambda \mathbf{x}_\alpha \mathbf{A}_o]$ .
$\wedge_{ooo}$	stands for	$[\lambda x_o \lambda y_o [[\lambda g_{ooo} [g_{ooo} T_o T_o]] = [\lambda g_{ooo} [g_{ooo} x_o y_o]]]]$ .
$[\mathbf{A}_o \wedge \mathbf{B}_o]$	stands for	$[\wedge_{ooo} \mathbf{A}_o \mathbf{B}_o]$ .
$\supset_{ooo}$	stands for	$[\lambda x_o \lambda y_o [x_o = [x_o \wedge y_o]]]$ .
$[\mathbf{A}_o \supset \mathbf{B}_o]$	stands for	$[\supset_{ooo} \mathbf{A}_o \mathbf{B}_o]$ .
$\sim_{oo}$	stands for	$[\mathbf{Q}_{ooo} F_o]$ .
$[\sim \mathbf{A}_o]$	stands for	$[\sim_{oo} \mathbf{A}_o]$ .
$\vee_{ooo}$	stands for	$[\lambda x_o \lambda y_o [\sim [[\sim x_o] \wedge [\sim y_o]]]]$ .
$[\mathbf{A}_o \vee \mathbf{B}_o]$	stands for	$[\vee_{ooo} \mathbf{A}_o \mathbf{B}_o]$ .
$[\exists \mathbf{x}_\alpha \mathbf{A}_o]$	stands for	$[\sim [\forall \mathbf{x}_\alpha \sim \mathbf{A}_o]]$ .
$[\exists_1 \mathbf{x}_\alpha \mathbf{A}_o]$	stands for	$[\exists \mathbf{x}_\alpha [[\lambda \mathbf{x}_\alpha \mathbf{A}_o] = \mathbf{Q}_{o\alpha\alpha} \mathbf{x}_\alpha]]$ .
$[\mathbf{A}_\alpha \neq \mathbf{B}_\alpha]$	stands for	$[\sim [\mathbf{A}_\alpha = \mathbf{B}_\alpha]]$ .
$[\mathbf{A}_\alpha \downarrow]$	stands for	$[\mathbf{A}_\alpha = \mathbf{A}_\alpha]$ .
$[\mathbf{A}_\alpha \uparrow]$	stands for	$[\sim [\mathbf{A}_\alpha \downarrow]]$ .
$[\mathbf{A}_\alpha \simeq \mathbf{B}_\alpha]$	stands for	$[\mathbf{A}_\alpha \downarrow \vee \mathbf{B}_\alpha \downarrow] \supset [\mathbf{A}_\alpha = \mathbf{B}_\alpha]$ .
$[\mathbf{I}x_\alpha \mathbf{A}_o]$	stands for	$[\iota_{\alpha(o\alpha)} [\lambda \mathbf{x}_\alpha \mathbf{A}_o]]$ where $\alpha \neq o$ .
$\perp_o$	stands for	$F_o$ .
$\perp_\alpha$	stands for	$[\mathbf{I}x_\alpha [x_\alpha \neq x_\alpha]]$ where $\alpha \neq o$ .
$[\text{if } \mathbf{A}_o \mathbf{B}_\alpha \mathbf{C}_\alpha]$	stands for	$[\mathbf{c}\mathbf{A}_o \mathbf{B}_\alpha \mathbf{C}_\alpha]$ .
$\ulcorner \mathbf{A}_\alpha \urcorner$	stands for	$[\mathbf{q}\mathbf{A}_\alpha]$ .
$[\mathbf{A}_\epsilon]_\alpha$	stands for	$[\mathbf{e}\mathbf{A}_\epsilon x_\alpha]$ .
$\text{fst}_{\langle \alpha \beta \rangle}$	stands for	$\lambda z_{\langle \alpha \beta \rangle} \mathbf{I}x_\alpha \exists y_\beta [z_{\langle \alpha \beta \rangle} = \text{pair}_{\langle \alpha \beta \rangle \beta \alpha} x_\alpha y_\beta]$ .
$\text{snd}_{\langle \beta \alpha \rangle}$	stands for	$\lambda z_{\langle \alpha \beta \rangle} \mathbf{I}y_\beta \exists x_\alpha [z_{\langle \alpha \beta \rangle} = \text{pair}_{\langle \alpha \beta \rangle \beta \alpha} x_\alpha y_\beta]$ .
$\text{var}_{(o\epsilon)}^\alpha$	stands for	$\lambda x_\epsilon [\text{var}_{(o\epsilon)} x_\epsilon \wedge \text{wff}_{(o\epsilon)}^\alpha x_\epsilon]$ .
$\text{con}_{(o\epsilon)}^\alpha$	stands for	$\lambda x_\epsilon [\text{con}_{(o\epsilon)} x_\epsilon \wedge \text{wff}_{(o\epsilon)}^\alpha x_\epsilon]$ .
$\text{eval-free}_{(o\epsilon)}^\alpha$	stands for	$\lambda x_\epsilon [\text{eval-free}_{o\epsilon} x_\epsilon \wedge \text{wff}_{o\epsilon}^\alpha x_\epsilon]$ .
$\text{syn-closed}_{(o\epsilon)}$	stands for	$\lambda x_\epsilon \forall y_\epsilon [\text{var}_{o\epsilon} y_\epsilon \supset \text{not-free-in}_{o\epsilon\epsilon} y_\epsilon x_\epsilon]$ .

Table 3: Definitions and Abbreviations



**Theorem 5.0.5 (Replete Syntax Framework)**  $F = (\mathcal{D}_\epsilon \cup \{\perp\}, \mathcal{E}, \mathcal{L}_\epsilon, Q, E)$  is a replete syntax framework for  $(\mathcal{L}, I)$ .

**Proof**  $F$  is a syntax framework since it satisfies the following conditions:

1.  $R$  is a syntax representation of  $\mathcal{L}$  by Proposition 5.0.3.
2.  $(\mathcal{L}_\epsilon, I)$  is a syntax language for  $R$  by Proposition 5.0.4.
3. For all  $\ulcorner \mathbf{D}_\delta \urcorner \in \mathcal{L}$ ,

$$\mathcal{W}_\varphi^{\mathcal{M}}(Q(\mathbf{D}_\delta)) = \mathcal{W}_\varphi^{\mathcal{M}}(\ulcorner \mathbf{D}_\delta \urcorner) = \mathcal{V}_\varphi^{\mathcal{M}}(\ulcorner \mathbf{D}_\delta \urcorner) = \mathcal{E}(\mathbf{D}_\delta),$$

i.e., the Quotation Axiom holds.

4. For all  $\ulcorner \mathbf{A}_\epsilon \urcorner \in \mathcal{L}_\epsilon$ ,

$$\begin{aligned} & \mathcal{W}_\varphi^{\mathcal{M}}(E(\mathbf{A}_\epsilon)) \\ &= \mathcal{W}_\varphi^{\mathcal{M}}(\llbracket \mathbf{A}_\epsilon \rrbracket_\alpha) \\ &= \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \mathbf{A}_\epsilon \rrbracket_\alpha) \\ &= \mathcal{V}_\varphi^{\mathcal{M}}(\mathcal{E}^{-1}(\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\epsilon))) \\ &= \mathcal{W}_\varphi^{\mathcal{M}}(\mathcal{E}^{-1}(\mathcal{W}_\varphi^{\mathcal{M}}(\mathbf{A}_\epsilon))) \end{aligned}$$

if  $E(\mathbf{A}_\epsilon)$  is defined, i.e., the Evaluation Axiom holds.

Finally,  $F$  is replete since  $\mathcal{L}$  is both the object and full language of  $F$  and  $F$  has build-in quotation and evaluation.  $\square$

## 6 Normal Models

### 6.1 Specifications

In a general or evaluation-free model, the first three logical constants are specified as part of the definition of an interpretation, but the remaining 12 logical constants, which involve the type  $\epsilon$ , are not specified. In this section, each of these latter logical constants is specified below via a set of formulas called *specifying axioms*. Formula schemas are used to present the specifying axioms.

#### Specification 1 (Quotation)

$$\ulcorner \mathbf{A}_\alpha \urcorner = \mathcal{E}(\mathbf{A}_\alpha).$$

#### Specification 2 ( $\text{var}_{o\epsilon}$ )

1.  $\text{var}_{o\epsilon} \ulcorner \mathbf{x}_\alpha \urcorner$ .
2.  $\sim[\text{var}_{o\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner]$  where  $\mathbf{A}_\alpha$  is not a variable.

**Specification 3** ( $\text{con}_{o\epsilon}$ )

1.  $\text{con}_{o\epsilon} \ulcorner \mathbf{c}_\alpha \urcorner$  where  $\mathbf{c}_\alpha$  is a primitive constant.
2.  $\sim[\text{con}_{o\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner]$  where  $\mathbf{A}_\alpha$  is not a primitive constant.

**Specification 4** ( $\epsilon$ )

1.  $\sim[\text{var}_{o\epsilon} \mathbf{A}_\epsilon \wedge \text{con}_{o\epsilon} \mathbf{A}_\epsilon]$ .
2.  $\sim[\text{var}_{o\epsilon} \mathbf{A}_\epsilon \wedge \mathbf{A}_\epsilon = \text{app}_{\epsilon\epsilon\epsilon} \mathbf{D}_\epsilon \mathbf{E}_\epsilon]$ .
3.  $\sim[\text{var}_{o\epsilon} \mathbf{A}_\epsilon \wedge \mathbf{A}_\epsilon = \text{abs}_{\epsilon\epsilon\epsilon} \mathbf{D}_\epsilon \mathbf{E}_\epsilon]$ .
4.  $\sim[\text{var}_{o\epsilon} \mathbf{A}_\epsilon \wedge \mathbf{A}_\epsilon = \text{cond}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{D}_\epsilon \mathbf{E}_\epsilon \mathbf{F}_\epsilon]$ .
5.  $\sim[\text{var}_{o\epsilon} \mathbf{A}_\epsilon \wedge \mathbf{A}_\epsilon = \text{quot}_{\epsilon\epsilon} \mathbf{D}_\epsilon]$ .
6.  $\sim[\text{var}_{o\epsilon} \mathbf{A}_\epsilon \wedge \mathbf{A}_\epsilon = \text{eval}_{\epsilon\epsilon\epsilon} \mathbf{D}_\epsilon \mathbf{E}_\epsilon]$ .
7.  $\sim[\text{con}_{o\epsilon} \mathbf{A}_\epsilon \wedge \mathbf{A}_\epsilon = \text{app}_{\epsilon\epsilon\epsilon} \mathbf{D}_\epsilon \mathbf{E}_\epsilon]$ .
8.  $\sim[\text{con}_{o\epsilon} \mathbf{A}_\epsilon \wedge \mathbf{A}_\epsilon = \text{abs}_{\epsilon\epsilon\epsilon} \mathbf{D}_\epsilon \mathbf{E}_\epsilon]$ .
9.  $\sim[\text{con}_{o\epsilon} \mathbf{A}_\epsilon \wedge \mathbf{A}_\epsilon = \text{cond}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{D}_\epsilon \mathbf{E}_\epsilon \mathbf{F}_\epsilon]$ .
10.  $\sim[\text{con}_{o\epsilon} \mathbf{A}_\epsilon \wedge \mathbf{A}_\epsilon = \text{quot}_{\epsilon\epsilon} \mathbf{D}_\epsilon]$ .
11.  $\sim[\text{con}_{o\epsilon} \mathbf{A}_\epsilon \wedge \mathbf{A}_\epsilon = \text{eval}_{\epsilon\epsilon\epsilon} \mathbf{D}_\epsilon \mathbf{E}_\epsilon]$ .
12.  $\text{app}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon \neq \text{abs}_{\epsilon\epsilon\epsilon} \mathbf{D}_\epsilon \mathbf{E}_\epsilon$ .
13.  $\text{app}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon \neq \text{cond}_{\epsilon\epsilon\epsilon} \mathbf{D}_\epsilon \mathbf{E}_\epsilon \mathbf{F}_\epsilon$ .
14.  $\text{app}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon \neq \text{quot}_{\epsilon\epsilon} \mathbf{D}_\epsilon$ .
15.  $\text{app}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon \neq \text{eval}_{\epsilon\epsilon\epsilon} \mathbf{D}_\epsilon \mathbf{E}_\epsilon$ .
16.  $\text{abs}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon \neq \text{cond}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{D}_\epsilon \mathbf{E}_\epsilon \mathbf{F}_\epsilon$ .
17.  $\text{abs}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon \neq \text{quot}_{\epsilon\epsilon} \mathbf{D}_\epsilon$ .
18.  $\text{abs}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon \neq \text{eval}_{\epsilon\epsilon\epsilon} \mathbf{D}_\epsilon \mathbf{E}_\epsilon$ .
19.  $\text{cond}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon \mathbf{C}_\epsilon \neq \text{quot}_{\epsilon\epsilon} \mathbf{D}_\epsilon$ .
20.  $\text{cond}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon \mathbf{C}_\epsilon \neq \text{eval}_{\epsilon\epsilon\epsilon} \mathbf{D}_\epsilon \mathbf{E}_\epsilon$ .
21.  $\text{quot}_{\epsilon\epsilon} \mathbf{A}_\epsilon \neq \text{eval}_{\epsilon\epsilon\epsilon} \mathbf{D}_\epsilon \mathbf{E}_\epsilon$ .
22.  $\ulcorner \mathbf{x}_\alpha \urcorner \neq \ulcorner \mathbf{y}_\beta \urcorner$  where  $\mathbf{x}_\alpha \neq \mathbf{y}_\alpha$ .
23.  $\ulcorner \mathbf{c}_\alpha \urcorner \neq \ulcorner \mathbf{d}_\beta \urcorner$  where  $\mathbf{c}_\alpha$  and  $\mathbf{d}_\alpha$  are different primitive constants.

24.  $\text{app}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon = \text{app}_{\epsilon\epsilon\epsilon} \mathbf{D}_\epsilon \mathbf{E}_\epsilon \supset [\mathbf{A}_\epsilon = \mathbf{D}_\epsilon \wedge \mathbf{B}_\epsilon = \mathbf{E}_\epsilon]$ .
25.  $\text{abs}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon = \text{abs}_{\epsilon\epsilon\epsilon} \mathbf{D}_\epsilon \mathbf{E}_\epsilon \supset [\mathbf{A}_\epsilon = \mathbf{D}_\epsilon \wedge \mathbf{B}_\epsilon = \mathbf{E}_\epsilon]$ .
26.  $\text{cond}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon \mathbf{C}_\epsilon = \text{cond}_{\epsilon\epsilon\epsilon} \mathbf{D}_\epsilon \mathbf{E}_\epsilon \mathbf{F}_\epsilon \supset$   
 $[\mathbf{A}_\epsilon = \mathbf{D}_\epsilon \wedge \mathbf{B}_\epsilon = \mathbf{E}_\epsilon \wedge \mathbf{C}_\epsilon = \mathbf{F}_\epsilon]$ .
27.  $\text{quot}_{\epsilon\epsilon} \mathbf{A}_\epsilon = \text{quot}_{\epsilon\epsilon} \mathbf{D}_\epsilon \supset \mathbf{A}_\epsilon = \mathbf{D}_\epsilon$ .
28.  $\text{eval}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon = \text{eval}_{\epsilon\epsilon\epsilon} \mathbf{D}_\epsilon \mathbf{E}_\epsilon \supset [\mathbf{A}_\epsilon = \mathbf{D}_\epsilon \wedge \mathbf{B}_\epsilon = \mathbf{E}_\epsilon]$ .
29.  $[\mathbf{A}_o^1 \wedge \mathbf{A}_o^2 \wedge \mathbf{A}_o^3 \wedge \mathbf{A}_o^4 \wedge \mathbf{A}_o^5 \wedge \mathbf{A}_o^6 \wedge \mathbf{A}_o^7] \supset \forall x_\epsilon [p_{o\epsilon} x_\epsilon]$  where:
  - $\mathbf{A}_o^1$  is  $\forall x_\epsilon [\text{var}_{o\epsilon} x_\epsilon \supset p_{o\epsilon} x_\epsilon]$ .
  - $\mathbf{A}_o^2$  is  $\forall x_\epsilon [\text{con}_{o\epsilon} x_\epsilon \supset p_{o\epsilon} x_\epsilon]$ .
  - $\mathbf{A}_o^3$  is  $\forall x_\epsilon \forall y_\epsilon [[p_{o\epsilon} x_\epsilon \wedge p_{o\epsilon} y_\epsilon \wedge [\text{app}_{\epsilon\epsilon\epsilon} x_\epsilon y_\epsilon] \downarrow] \supset p_{o\epsilon} [\text{app}_{\epsilon\epsilon\epsilon} x_\epsilon y_\epsilon]]$ .
  - $\mathbf{A}_o^4$  is  $\forall x_\epsilon \forall y_\epsilon [[p_{o\epsilon} x_\epsilon \wedge p_{o\epsilon} y_\epsilon \wedge [\text{abs}_{\epsilon\epsilon\epsilon} x_\epsilon y_\epsilon] \downarrow] \supset p_{o\epsilon} [\text{abs}_{\epsilon\epsilon\epsilon} x_\epsilon y_\epsilon]]$ .
  - $\mathbf{A}_o^5$  is  $\forall x_\epsilon \forall y_\epsilon \forall z_\epsilon [[p_{o\epsilon} x_\epsilon \wedge p_{o\epsilon} y_\epsilon \wedge p_{o\epsilon} z_\epsilon \wedge [\text{cond}_{\epsilon\epsilon\epsilon\epsilon} x_\epsilon y_\epsilon z_\epsilon] \downarrow] \supset$   
 $p_{o\epsilon} [\text{cond}_{\epsilon\epsilon\epsilon\epsilon} x_\epsilon y_\epsilon z_\epsilon]]$ .
  - $\mathbf{A}_o^6$  is  $\forall x_\epsilon [p_{o\epsilon} x_\epsilon \supset p_{o\epsilon} [\text{quot}_{\epsilon\epsilon} x_\epsilon]]$ .
  - $\mathbf{A}_o^7$  is  $\forall x_\epsilon \forall y_\epsilon [[p_{o\epsilon} x_\epsilon \wedge p_{o\epsilon} y_\epsilon \wedge [\text{eval}_{\epsilon\epsilon\epsilon} x_\epsilon y_\epsilon] \downarrow] \supset p_{o\epsilon} [\text{eval}_{\epsilon\epsilon\epsilon} x_\epsilon y_\epsilon]]$ .

**Specification 5 (eval-free<sub>oε</sub>)**

1.  $\text{var}_{o\epsilon} \mathbf{A}_\epsilon \supset \text{eval-free}_{o\epsilon} \mathbf{A}_\epsilon$ .
2.  $\text{con}_{o\epsilon} \mathbf{A}_\epsilon \supset \text{eval-free}_{o\epsilon} \mathbf{A}_\epsilon$ .
3.  $[\text{app}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon] \downarrow \supset$   
 $\text{eval-free}_{o\epsilon} [\text{app}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon] \equiv [\text{eval-free}_{o\epsilon} \mathbf{A}_\epsilon \wedge \text{eval-free}_{o\epsilon} \mathbf{B}_\epsilon]$ .
4.  $[\text{abs}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon] \downarrow \supset$   
 $\text{eval-free}_{o\epsilon} [\text{abs}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon] \equiv \text{eval-free}_{o\epsilon} \mathbf{B}_\epsilon$ .
5.  $[\text{cond}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon \mathbf{C}_\epsilon] \downarrow \supset$   
 $\text{eval-free}_{o\epsilon} [\text{cond}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon \mathbf{C}_\epsilon] \equiv$   
 $[\text{eval-free}_{o\epsilon} \mathbf{A}_\epsilon \wedge \text{eval-free}_{o\epsilon} \mathbf{B}_\epsilon \wedge \text{eval-free}_{o\epsilon} \mathbf{C}_\epsilon]$ .
6.  $\mathbf{A}_\epsilon \downarrow \supset \text{eval-free}_{o\epsilon} [\text{quot}_{\epsilon\epsilon} \mathbf{A}_\epsilon]$ .
7.  $\sim [\text{eval-free}_{o\epsilon} [\text{eval}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon]]$ .

**Specification 6 (wff<sup>α</sup><sub>oε</sub>)**

1.  $\text{wff}_{o\epsilon}^\alpha \ulcorner \mathbf{x}_\alpha \urcorner$ .
2.  $\text{wff}_{o\epsilon}^\alpha \ulcorner \mathbf{c}_\alpha \urcorner$  where  $\mathbf{c}_\alpha$  is a primitive constant.
3.  $[\text{wff}_{o\epsilon}^{\alpha\beta} \mathbf{A}_\epsilon \wedge \text{wff}_{o\epsilon}^\beta \mathbf{B}_\epsilon] \supset \text{wff}_{o\epsilon}^\alpha [\text{app}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon]$ .

4.  $[\text{wff}_{o\epsilon}^t \mathbf{A}_\epsilon \vee \text{wff}_{o\epsilon}^o \mathbf{A}_\epsilon \vee \text{wff}_{o\epsilon}^\epsilon \mathbf{A}_\epsilon \vee \text{wff}_{o\epsilon}^{(\alpha\beta)} \mathbf{A}_\epsilon] \supset [\text{app}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon] \uparrow$ .
5.  $[\text{wff}_{o\epsilon}^{\alpha\beta} \mathbf{A}_\epsilon \wedge \sim[\text{wff}_{o\epsilon}^\beta \mathbf{B}_\epsilon]] \supset [\text{app}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon] \uparrow$ .
6.  $[\text{var}_{o\epsilon}^\alpha \mathbf{A}_\epsilon \wedge \text{wff}_{o\epsilon}^\beta \mathbf{B}_\epsilon] \supset \text{wff}_{o\epsilon}^{\beta\alpha} [\text{abs}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon]$ .
7.  $\sim[\text{var}_{o\epsilon} \mathbf{A}_\epsilon] \supset [\text{abs}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon] \uparrow$ .
8.  $[\text{wff}_{o\epsilon}^o \mathbf{A}_\epsilon \wedge \text{wff}_{o\epsilon}^\alpha \mathbf{B}_\epsilon \wedge \text{wff}_{o\epsilon}^\alpha \mathbf{C}_\epsilon] \supset \text{wff}_{o\epsilon}^\alpha [\text{cond}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon \mathbf{C}_\epsilon]$ .
9.  $[\sim[\text{wff}_{o\epsilon}^o \mathbf{A}_\epsilon] \vee [\text{wff}_{o\epsilon}^\alpha \mathbf{B}_\epsilon \wedge \text{wff}_{o\epsilon}^\beta \mathbf{C}_\epsilon]] \supset [\text{cond}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon \mathbf{C}_\epsilon] \uparrow$   
where  $\alpha \neq \beta$ .
10.  $\mathbf{A}_\epsilon \downarrow \supset \text{wff}_{o\epsilon}^c [\text{quot}_{\epsilon\epsilon} \mathbf{A}_\epsilon]$ .
11.  $[\text{wff}_{o\epsilon}^\epsilon \mathbf{A}_\epsilon \wedge \text{var}_{o\epsilon}^\alpha \mathbf{B}_\epsilon] \supset \text{wff}_{o\epsilon}^\alpha [\text{eval}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon]$ .
12.  $[\sim[\text{wff}_{o\epsilon}^\epsilon \mathbf{A}_\epsilon] \vee \sim[\text{var}_{o\epsilon} \mathbf{B}_\epsilon]] \supset [\text{eval}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon] \uparrow$ .
13.  $\sim[\text{wff}_{o\epsilon}^\alpha \mathbf{A}_\epsilon \wedge \text{wff}_{o\epsilon}^\beta \mathbf{A}_\epsilon]$  where  $\alpha \neq \beta$ .

**Specification 7** (not-free-in<sub>oεε</sub>)

1.  $\text{var}_{o\epsilon} \mathbf{A}_\epsilon \supset \sim[\text{not-free-in}_{o\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{A}_\epsilon]$ .
2.  $[\text{var}_{o\epsilon} \mathbf{A}_\epsilon \wedge \text{var}_{o\epsilon} \mathbf{B}_\epsilon \wedge \mathbf{A}_\epsilon \neq \mathbf{B}_\epsilon] \supset \text{not-free-in}_{o\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon$ .
3.  $[\text{var}_{o\epsilon} \mathbf{A}_\epsilon \wedge \text{con}_{o\epsilon} \mathbf{B}_\epsilon] \supset \text{not-free-in}_{o\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon$ .
4.  $[\text{var}_{o\epsilon} \mathbf{A}_\epsilon \wedge [\text{app}_{\epsilon\epsilon\epsilon} \mathbf{B}_\epsilon \mathbf{C}_\epsilon] \downarrow] \supset$   
 $\text{not-free-in}_{o\epsilon\epsilon} \mathbf{A}_\epsilon [\text{app}_{\epsilon\epsilon\epsilon} \mathbf{B}_\epsilon \mathbf{C}_\epsilon] \equiv$   
 $[\text{not-free-in}_{o\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon \wedge \text{not-free-in}_{o\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{C}_\epsilon]$ .
5.  $[\text{var}_{o\epsilon} \mathbf{A}_\epsilon \wedge [\text{abs}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon] \downarrow] \supset \text{not-free-in}_{o\epsilon\epsilon} \mathbf{A}_\epsilon [\text{abs}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon]$ .
6.  $[\text{var}_{o\epsilon} \mathbf{A}_\epsilon \wedge \text{var}_{o\epsilon} \mathbf{B}_\epsilon \wedge \mathbf{A}_\epsilon \neq \mathbf{B}_\epsilon \wedge [\text{abs}_{\epsilon\epsilon\epsilon} \mathbf{B}_\epsilon \mathbf{C}_\epsilon] \downarrow] \supset$   
 $\text{not-free-in}_{o\epsilon\epsilon} \mathbf{A}_\epsilon [\text{abs}_{\epsilon\epsilon\epsilon} \mathbf{B}_\epsilon \mathbf{C}_\epsilon] \equiv \text{not-free-in}_{o\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{C}_\epsilon$ .
7.  $[\text{var}_{o\epsilon} \mathbf{A}_\epsilon \wedge [\text{cond}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{D}_\epsilon \mathbf{E}_\epsilon \mathbf{F}_\epsilon] \downarrow] \supset$   
 $\text{not-free-in}_{o\epsilon\epsilon} \mathbf{A}_\epsilon [\text{cond}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{D}_\epsilon \mathbf{E}_\epsilon \mathbf{F}_\epsilon] \equiv$   
 $[\text{not-free-in}_{o\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{D}_\epsilon \wedge \text{not-free-in}_{o\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{E}_\epsilon \wedge \text{not-free-in}_{o\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{F}_\epsilon]$ .
8.  $[\text{var}_{o\epsilon} \mathbf{A}_\epsilon \wedge \mathbf{B}_\epsilon \downarrow] \supset \text{not-free-in}_{o\epsilon\epsilon} \mathbf{A}_\epsilon [\text{quot}_{\epsilon\epsilon} \mathbf{B}_\epsilon]$ .
9.  $[\text{var}_{o\epsilon} \mathbf{A}_\epsilon \wedge \text{var}_{o\epsilon}^\alpha \mathbf{C}_\epsilon \wedge [\text{eval}_{\epsilon\epsilon\epsilon} \mathbf{B}_\epsilon \mathbf{C}_\epsilon] \downarrow] \supset$   
 $\text{not-free-in}_{o\epsilon\epsilon} \mathbf{A}_\epsilon [\text{eval}_{\epsilon\epsilon\epsilon} \mathbf{B}_\epsilon \mathbf{C}_\epsilon] \equiv$   
 $[\text{syn-closed}_{o\epsilon} \mathbf{B}_\epsilon \wedge \text{eval-free}_{o\epsilon}^\epsilon \mathbf{B}_\epsilon \wedge$   
 $\text{eval-free}_{o\epsilon}^\alpha [\mathbf{B}_\epsilon]_\epsilon \wedge \text{not-free-in}_{o\epsilon\epsilon} \mathbf{A}_\epsilon [\mathbf{B}_\epsilon]_\epsilon]$ .
10.  $\sim[\text{var}_{o\epsilon} \mathbf{A}_\epsilon] \supset \text{not-free-in}_{o\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon$ .

**Specification 8** ( $\text{cleanse}_{\epsilon\epsilon}$ )

1.  $\text{var}_{o\epsilon} \mathbf{A}_\epsilon \supset \text{cleanse}_{\epsilon\epsilon} \mathbf{A}_\epsilon = \mathbf{A}_\epsilon.$
2.  $\text{con}_{o\epsilon} \mathbf{A}_\epsilon \supset \text{cleanse}_{\epsilon\epsilon} \mathbf{A}_\epsilon = \mathbf{A}_\epsilon.$
3.  $[\text{app}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon] \downarrow \supset$   
 $\text{cleanse}_{\epsilon\epsilon} [\text{app}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon] \simeq \text{app}_{\epsilon\epsilon\epsilon} [\text{cleanse}_{\epsilon\epsilon} \mathbf{A}_\epsilon] [\text{cleanse}_{\epsilon\epsilon} \mathbf{B}_\epsilon].$
4.  $[\text{abs}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon] \downarrow \supset$   
 $\text{cleanse}_{\epsilon\epsilon} [\text{abs}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon] \simeq \text{abs}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon [\text{cleanse}_{\epsilon\epsilon} \mathbf{B}_\epsilon].$
5.  $[\text{cond}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon \mathbf{C}_\epsilon] \downarrow \supset$   
 $\text{cleanse}_{\epsilon\epsilon} [\text{cond}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon \mathbf{C}_\epsilon] \simeq$   
 $\text{cond}_{\epsilon\epsilon\epsilon\epsilon} [\text{cleanse}_{\epsilon\epsilon} \mathbf{A}_\epsilon] [\text{cleanse}_{\epsilon\epsilon} \mathbf{B}_\epsilon] [\text{cleanse}_{\epsilon\epsilon} \mathbf{C}_\epsilon].$
6.  $\text{cleanse}_{\epsilon\epsilon} [\text{quot}_{\epsilon\epsilon} \mathbf{A}_\epsilon] \simeq [\text{quot}_{\epsilon\epsilon} \mathbf{A}_\epsilon].$
7.  $[\text{var}_{o\epsilon}^\alpha \mathbf{B}_\epsilon \wedge [\text{eval}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon] \downarrow] \supset$   
 $\text{cleanse}_{\epsilon\epsilon} [\text{eval}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon] \simeq$   
 if  $[\text{syn-closed}_{o\epsilon} \mathbf{E}_\epsilon \wedge \text{eval-free}_{o\epsilon}^\alpha [\mathbf{E}_\epsilon]_\epsilon] [\mathbf{E}_\epsilon]_\epsilon \perp_\epsilon$   
 where  $\mathbf{E}_\epsilon$  is  $[\text{cleanse}_{\epsilon\epsilon} \mathbf{A}_\epsilon].$

**Specification 9** ( $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$ )

1.  $[\text{wff}_{o\epsilon}^\alpha \mathbf{A}_\epsilon \wedge \text{var}_{o\epsilon}^\alpha \mathbf{B}_\epsilon] \supset$   
 $\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon \mathbf{B}_\epsilon = \text{cleanse}_{\epsilon\epsilon} \mathbf{A}_\epsilon.$
2.  $[\text{wff}_{o\epsilon}^\alpha \mathbf{A}_\epsilon \wedge \text{var}_{o\epsilon}^\alpha \mathbf{B}_\epsilon \wedge \text{var}_{o\epsilon} \mathbf{C}_\epsilon \wedge \mathbf{B}_\epsilon \neq \mathbf{C}_\epsilon] \supset$   
 $\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon \mathbf{C}_\epsilon = \mathbf{C}_\epsilon.$
3.  $[\text{wff}_{o\epsilon}^\alpha \mathbf{A}_\epsilon \wedge \text{var}_{o\epsilon}^\alpha \mathbf{B}_\epsilon \wedge \text{con}_{o\epsilon} \mathbf{C}_\epsilon] \supset$   
 $\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon \mathbf{C}_\epsilon = \mathbf{C}_\epsilon.$
4.  $[\text{wff}_{o\epsilon}^\alpha \mathbf{A}_\epsilon \wedge \text{var}_{o\epsilon}^\alpha \mathbf{B}_\epsilon \wedge [\text{app}_{\epsilon\epsilon\epsilon} \mathbf{D}_\epsilon \mathbf{E}_\epsilon] \downarrow] \supset$   
 $\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon [\text{app}_{\epsilon\epsilon\epsilon} \mathbf{D}_\epsilon \mathbf{E}_\epsilon] \simeq$   
 $\text{app}_{\epsilon\epsilon\epsilon} [\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon \mathbf{D}_\epsilon] [\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon \mathbf{E}_\epsilon].$
5.  $[\text{wff}_{o\epsilon}^\alpha \mathbf{A}_\epsilon \wedge \text{var}_{o\epsilon}^\alpha \mathbf{B}_\epsilon \wedge [\text{abs}_{\epsilon\epsilon\epsilon} \mathbf{B}_\epsilon \mathbf{E}_\epsilon] \downarrow] \supset$   
 $\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon [\text{abs}_{\epsilon\epsilon\epsilon} \mathbf{B}_\epsilon \mathbf{E}_\epsilon] \simeq \text{abs}_{\epsilon\epsilon\epsilon} \mathbf{B}_\epsilon [\text{cleanse}_{\epsilon\epsilon} \mathbf{A}_\epsilon].$
6.  $[\text{wff}_{o\epsilon}^\alpha \mathbf{A}_\epsilon \wedge \text{var}_{o\epsilon}^\alpha \mathbf{B}_\epsilon \wedge \text{var}_{o\epsilon} \mathbf{D}_\epsilon \wedge \mathbf{B}_\epsilon \neq \mathbf{D}_\epsilon \wedge [\text{abs}_{\epsilon\epsilon\epsilon} \mathbf{D}_\epsilon \mathbf{E}_\epsilon] \downarrow] \supset$   
 $\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon [\text{abs}_{\epsilon\epsilon\epsilon} \mathbf{D}_\epsilon \mathbf{E}_\epsilon] \simeq$   
 if  $[\text{not-free-in}_{o\epsilon\epsilon} \mathbf{B}_\epsilon \mathbf{E}_\epsilon \vee \text{not-free-in}_{o\epsilon\epsilon} \mathbf{D}_\epsilon \mathbf{A}_\epsilon]$   
 $[\text{abs}_{\epsilon\epsilon\epsilon} \mathbf{D}_\epsilon [\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon \mathbf{E}_\epsilon]]$   
 $\perp_\epsilon.$

7.  $[\text{wff}_{o\epsilon}^\alpha \mathbf{A}_\epsilon \wedge \text{var}_{o\epsilon}^\alpha \mathbf{B}_\epsilon \wedge [\text{cond}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{D}_\epsilon \mathbf{E}_\epsilon \mathbf{F}_\epsilon] \downarrow] \supset$   
 $\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon [\text{cond}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{D}_\epsilon \mathbf{E}_\epsilon \mathbf{F}_\epsilon] \simeq$   
 $\text{cond}_{\epsilon\epsilon\epsilon\epsilon} [\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon \mathbf{D}_\epsilon] [\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon \mathbf{E}_\epsilon] [\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon \mathbf{F}_\epsilon].$
8.  $[\text{wff}_{o\epsilon}^\alpha \mathbf{A}_\epsilon \wedge \text{var}_{o\epsilon}^\alpha \mathbf{B}_\epsilon \wedge \mathbf{C}_\epsilon \downarrow] \supset$   
 $\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon [\text{quot}_{\epsilon\epsilon} \mathbf{C}_\epsilon] = \text{quot}_{\epsilon\epsilon} \mathbf{C}_\epsilon.$
9.  $[\text{wff}_{o\epsilon}^\alpha \mathbf{A}_\epsilon \wedge \text{var}_{o\epsilon}^\alpha \mathbf{B}_\epsilon \wedge \text{var}_{o\epsilon}^\beta \mathbf{E}_\epsilon \wedge [\text{eval}_{\epsilon\epsilon\epsilon} \mathbf{D}_\epsilon \mathbf{E}_\epsilon] \downarrow] \supset$   
 $\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon [\text{eval}_{\epsilon\epsilon\epsilon} \mathbf{D}_\epsilon \mathbf{E}_\epsilon] \simeq$   
 $\text{if} [\text{syn-closed}_{o\epsilon} \mathbf{E}_\epsilon^1 \wedge \text{eval-free}_{o\epsilon}^\beta \llbracket \mathbf{E}_\epsilon^1 \rrbracket_\epsilon] \mathbf{E}_\epsilon^2 \perp_\epsilon$   
 where:  
 $\mathbf{E}_\epsilon^1$  is  $[\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon \mathbf{D}_\epsilon].$   
 $\mathbf{E}_\epsilon^2$  is  $[\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon \llbracket \mathbf{E}_\epsilon^1 \rrbracket_\epsilon].$
10.  $[\text{wff}_{o\epsilon}^\alpha \mathbf{A}_\epsilon \wedge \sim[\text{var}_{o\epsilon}^\alpha \mathbf{B}_\epsilon] \supset$   
 $[\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon \mathbf{C}_\epsilon] \uparrow.$

## 6.2 Normal General and Evaluation-Free Models

Let  $\mathcal{S}$  be the total set of specifying axioms given above. A general model  $\mathcal{M}$  for  $\mathcal{Q}_0^{\text{uqe}}$  is *normal* if  $\mathcal{M} \models \mathbf{A}_o$  for all  $\mathbf{A}_o \in \mathcal{S}$ . We write  $\mathcal{H} \models_n \mathbf{A}_o$  to mean  $\mathcal{M} \models \mathbf{A}_o$  for every normal general model  $\mathcal{M}$  for  $\mathcal{H}$  where  $\mathcal{H}$  is a set of wffs $_o$ . We write  $\models_n \mathbf{A}_o$  to mean  $\emptyset \models_n \mathbf{A}_o$ .  $\mathbf{A}_o$  is *valid* in  $\mathcal{Q}_0^{\text{uqe}}$  if  $\models_n \mathbf{A}_o$ .

An evaluation-free model  $\mathcal{M}$  for  $\mathcal{Q}_0^{\text{uqe}}$  is *normal* if  $\mathcal{M} \models \mathbf{A}_o$  for all evaluation-free  $\mathbf{A}_o \in \mathcal{S}$ . We write  $\mathcal{H} \models_n^{\text{ef}} \mathbf{A}_o$  to mean  $\mathcal{M} \models \mathbf{A}_o$  for every normal evaluation-free model  $\mathcal{M}$  for  $\mathcal{H}$  where  $\mathbf{A}_o$  is evaluation-free and  $\mathcal{H}$  is a set of evaluation-free wffs $_o$ . We write  $\models_n^{\text{ef}} \mathbf{A}_o$  to mean  $\emptyset \models_n^{\text{ef}} \mathbf{A}_o$ .

Since standard models exist, normal general models (and hence normal evaluation-free models) exist by Corollary 6.2.3 given below.

**Proposition 6.2.1** *Let  $\mathcal{M}$  be a normal general model for  $\mathcal{Q}_0^{\text{uqe}}$ . Then  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathcal{E}(\mathbf{A}_\alpha)) = \mathcal{E}(\mathbf{A}_\alpha)$  for all  $\varphi \in \text{assign}(\mathcal{M})$  and  $\mathbf{A}_\alpha$ .*

**Proof** Immediate from the Specification 1 and the semantics of quotation.  $\square$

**Note 16 (Construction Literals)** The previous proposition says that a wff of the form  $\mathcal{E}(\mathbf{A}_\alpha)$  denotes itself. Thus each image of  $\mathcal{E}$  is a *literal*: its value is directly represented by its syntax. Quotation can be viewed as an operation that constructs literals for syntactic values. Florian Rabe explores in [56] a kind of quotation that constructs literals for semantic values.

**Note 17 (Quasiquotation)** Quasiquotation is a parameterized form of quotation in which the parameters serve as holes in a quotation that are filled with the values of expressions. It is a very powerful syntactic device for specifying expressions and defining macros. Quasiquotation was introduced by Willard Van Orman Quine in 1940 in the first version of his book *Mathematical Logic* [55]. It

has been extensively employed in the Lisp family of programming languages [4].<sup>2</sup> A *quasiquote* in  $\mathcal{Q}_0^{\text{uqe}}$  is a wff of the form  $\mathcal{E}(\mathbf{A}_\alpha)$  where some of its subwffs have been replaced by wffs $_\epsilon$ . As an example, suppose  $\mathbf{A}_\alpha$  is  $\wedge_{ooo} F_o T_o$  and so  $\mathcal{E}(\mathbf{A}_\alpha)$  is

$$\text{app}_{\epsilon\epsilon\epsilon} [\text{app}_{\epsilon\epsilon\epsilon} \ulcorner \wedge_{ooo} \urcorner \mathcal{E}(F_o)] \mathcal{E}(T_o).$$

Then

$$\text{app}_{\epsilon\epsilon\epsilon} [\text{app}_{\epsilon\epsilon\epsilon} \ulcorner \wedge_{ooo} \urcorner \mathbf{B}_\epsilon] \mathbf{C}_\epsilon$$

is a quasiquote that we will write in the more suggestive form

$$\ulcorner \wedge_{ooo} [\mathbf{B}_\epsilon] [\mathbf{C}_\epsilon] \urcorner.$$

$[\mathbf{B}_\epsilon]$  and  $[\mathbf{C}_\epsilon]$  are holes in the quotation  $\ulcorner \mathbf{A}_\alpha \urcorner$  that are filled with the values of  $\mathbf{B}_\epsilon$  and  $\mathbf{C}_\epsilon$ . For instance, if  $\mathbf{B}_\epsilon$  and  $\mathbf{C}_\epsilon$  are  $\ulcorner \mathbf{D}_o \urcorner$  and  $\ulcorner \mathbf{E}_o \urcorner$ , then

$$\ulcorner \wedge_{ooo} [\mathbf{B}_\epsilon] [\mathbf{C}_\epsilon] \urcorner = \ulcorner \wedge_{ooo} [\ulcorner \mathbf{D}_o \urcorner] [\ulcorner \mathbf{E}_o \urcorner] \urcorner = \ulcorner \wedge_{ooo} \mathbf{D}_o \mathbf{E}_o \urcorner.$$

**Lemma 6.2.2** *Let  $\mathcal{M}$  be a standard model,  $\mathbf{c}_{\alpha_1}^1, \dots, \mathbf{c}_{\alpha_{11}}^{11}$  be the 11 logical constants  $\text{var}_{o\epsilon}$ ,  $\text{con}_{o\epsilon}$ ,  $\text{app}_{\epsilon\epsilon\epsilon}$ ,  $\text{abs}_{\epsilon\epsilon\epsilon}$ ,  $\text{cond}_{\epsilon\epsilon\epsilon\epsilon}$ ,  $\text{quot}_{\epsilon\epsilon}$ ,  $\text{eval}_{\epsilon\epsilon\epsilon}$ ,  $\text{eval-free}_{o\epsilon}$ ,  $\text{not-free-in}_{o\epsilon\epsilon}$ ,  $\text{cleanse}_{\epsilon\epsilon}$ , and  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$ , and  $\mathbf{d}_\beta^\alpha$  be the logical constant  $\text{wff}_{o\epsilon}^\alpha$  for each  $\alpha \in \mathcal{T}$ . Then there are unique functions  $f^1 \in \mathcal{D}_{\alpha_1}, \dots, f^{11} \in \mathcal{D}_{\alpha_{11}}$  and  $g^\alpha \in \mathcal{D}_\beta$  for each  $\alpha \in \mathcal{T}$  such that the members of  $\mathcal{S}$  are satisfied when  $\mathbf{c}_{\alpha_1}^1, \dots, \mathbf{c}_{\alpha_{11}}^{11}$  and  $\mathbf{d}_\beta^\alpha$  for each  $\alpha \in \mathcal{T}$  are interpreted in  $\mathcal{M}$  by  $f^1, \dots, f^{11}$  and  $g^\alpha$  for each  $\alpha \in \mathcal{T}$ , respectively.*

**Proof** Let  $\mathcal{M} = \langle \{\mathcal{D}_\alpha \mid \alpha \in \mathcal{T}\}, \mathcal{J} \rangle$  be a standard model for  $\mathcal{Q}_0^{\text{uqe}}$ . Then  $\mathcal{D}_\epsilon = \{\mathcal{E}(\mathbf{A}_\alpha) \mid \mathbf{A}_\alpha \text{ is a wff}\}$  by the Proposition 6.3.1 stated below.  $f^1$  is the predicate  $p \in \mathcal{D}_{o\epsilon}$  such that, for all wffs  $\mathbf{A}_\alpha$ ,  $p(\mathcal{E}(\mathbf{A}_\alpha)) = \text{T}$  iff  $\mathbf{A}_\alpha$  is a variable.  $f^2$  is the predicate  $p \in \mathcal{D}_{o\epsilon}$  such that, for all wffs  $\mathbf{A}_\alpha$ ,  $p(\mathcal{E}(\mathbf{A}_\alpha)) = \text{T}$  iff  $\mathbf{A}_\alpha$  is a primitive constant.

$f^3$  is the function  $f \in \mathcal{D}_{\epsilon\epsilon\epsilon}$  such that, for all wffs  $\mathbf{A}_\alpha$  and  $\mathbf{B}_\beta$ , if  $[\mathbf{A}_\alpha \mathbf{B}_\beta]$  is a wff, then  $f(\mathcal{E}(\mathbf{A}_\alpha))(\mathcal{E}(\mathbf{B}_\beta))$  is the wff  $[\text{app}_{\epsilon\epsilon\epsilon} \mathcal{E}(\mathbf{A}_\alpha) \mathcal{E}(\mathbf{B}_\beta)]$ , and otherwise  $f(\mathcal{E}(\mathbf{A}_\alpha))(\mathcal{E}(\mathbf{B}_\beta))$  is undefined.  $f^4$  is the function  $f \in \mathcal{D}_{\epsilon\epsilon\epsilon}$  such that, for all wffs  $\mathbf{A}_\alpha$  and  $\mathbf{B}_\beta$ , if  $[\lambda \mathbf{A}_\alpha \mathbf{B}_\beta]$  is a wff, then  $f(\mathcal{E}(\mathbf{A}_\alpha))(\mathcal{E}(\mathbf{B}_\beta))$  is the wff  $[\text{abs}_{\epsilon\epsilon\epsilon} \mathcal{E}(\mathbf{A}_\alpha) \mathcal{E}(\mathbf{B}_\beta)]$ , and otherwise  $f(\mathcal{E}(\mathbf{A}_\alpha))(\mathcal{E}(\mathbf{B}_\beta))$  is undefined.  $f^5$  is the function  $f \in \mathcal{D}_{\epsilon\epsilon\epsilon\epsilon}$  such that, for all wffs  $\mathbf{A}_o$ ,  $\mathbf{B}_\alpha$ , and  $\mathbf{C}_\alpha$ , if  $[\mathbf{c} \mathbf{A}_o \mathbf{B}_\alpha \mathbf{C}_\alpha]$  is a wff, then  $f(\mathcal{E}(\mathbf{A}_o))(\mathcal{E}(\mathbf{B}_\alpha))(\mathcal{E}(\mathbf{C}_\alpha))$  is the wff  $[\text{cond}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{A}_o \mathbf{B}_\alpha \mathbf{C}_\alpha]$ , and otherwise  $f(\mathcal{E}(\mathbf{A}_o))(\mathcal{E}(\mathbf{B}_\alpha))(\mathcal{E}(\mathbf{C}_\alpha))$  is undefined.  $f^6$  is the function  $f \in \mathcal{D}_{\epsilon\epsilon}$  such that, for all wffs  $\mathbf{A}_\alpha$ ,  $f(\mathcal{E}(\mathbf{A}_\alpha))$  is the wff  $[\text{quot}_{\epsilon\epsilon} \mathcal{E}(\mathbf{A}_\alpha)]$ .  $f^7$  is the function  $f \in \mathcal{D}_{\epsilon\epsilon\epsilon}$  such that, for all wffs  $\mathbf{A}_\alpha$  and  $\mathbf{B}_\beta$ , if  $[\mathbf{e} \mathbf{A}_\alpha \mathbf{B}_\beta]$  is a wff, then  $f(\mathcal{E}(\mathbf{A}_\alpha))(\mathcal{E}(\mathbf{B}_\beta))$  is the wff  $[\text{eval}_{\epsilon\epsilon\epsilon} \mathcal{E}(\mathbf{A}_\alpha) \mathcal{E}(\mathbf{B}_\beta)]$ , and otherwise  $f(\mathcal{E}(\mathbf{A}_\alpha))(\mathcal{E}(\mathbf{B}_\beta))$  is undefined.

$f^8$  is the predicate  $p \in \mathcal{D}_{o\epsilon}$  such that, for all wffs  $\mathbf{A}_\alpha$ ,  $p(\mathcal{E}(\mathbf{A}_\alpha)) = \text{T}$  iff  $\mathbf{A}_\alpha$  is evaluation-free. And, for each  $\alpha$ ,  $g^\alpha$  is the predicate  $p \in \mathcal{D}_{o\epsilon}$  such that, for

<sup>2</sup>In Lisp, the standard symbol for quasiquote is the backquote (‘) symbol, and thus in Lisp, quasiquote is usually called *backquote*.

all wffs  $\mathbf{A}_\beta$ ,  $p(\mathcal{E}(\mathbf{A}_\beta)) = \top$  iff  $\beta = \alpha$ . All of these functions above clearly satisfy the specifying axioms in  $\mathcal{S}$  that pertain to them.

$f^9$  is the unique function constructed by defining  $f^9(\mathcal{E}(\mathbf{A}_\alpha))(\mathcal{E}(\mathbf{B}_\beta))$  for all wffs  $\mathbf{A}_\alpha$  and  $\mathbf{B}_\beta$  by recursion on the complexity of  $\mathbf{B}_\beta$  in accordance with Specification 7.  $f^{10}$  and  $f^{11}$  are constructed similarly.  $\square$

**Corollary 6.2.3** *If  $\mathcal{M}$  is a standard model for  $\mathcal{Q}_0^{\text{uqe}}$ , then there is normal standard model  $\mathcal{M}'$  for  $\mathcal{Q}_0^{\text{uqe}}$  having the same frame as  $\mathcal{M}$ .*

A normal general model or evaluation-free model is a general or evaluation-free model  $\mathcal{M} = \langle \{\mathcal{D}_\alpha \mid \alpha \in \mathcal{T}\}, \mathcal{J} \rangle$  in which the structure of the domain  $\mathcal{D}_\epsilon$  is accessible via the logical constants involving  $\epsilon$ . From this point on, we will only be interested in general and evaluation-free models that are normal.

### 6.3 Nonstandard Constructions

Let  $\mathcal{M} = \langle \{\mathcal{D}_\alpha \mid \alpha \in \mathcal{T}\}, \mathcal{J} \rangle$  be a normal general model and  $d \in \mathcal{D}_\epsilon$ . The construction  $d$  is *standard* if  $d = \mathcal{E}(\mathbf{A}_\alpha)$  for some wff  $\mathbf{A}_\alpha$  and is *nonstandard* if it is not standard. That is, if  $d$  is nonstandard, then  $d \in \mathcal{D}_\epsilon \setminus \{\mathcal{E}(\mathbf{A}_\alpha) \mid \mathbf{A}_\alpha \text{ is a wff}\}$ .

One might think that Specification 4.29, the induction principle for the type  $\epsilon$ , would rule out the possibility of nonstandard constructions in  $\mathcal{M}$ . This is the case only when  $\mathcal{D}_{o_\epsilon}$  contains all possible predicates. Thus the following proposition holds:

**Proposition 6.3.1** *If  $\mathcal{M}$  is a normal standard model for  $\mathcal{Q}_0^{\text{uqe}}$ , then  $\mathcal{D}_\epsilon = \{\mathcal{E}(\mathbf{A}_\alpha) \mid \mathbf{A}_\alpha \text{ is a wff}\}$ , i.e.,  $\mathcal{M}$  contains no nonstandard constructions.*

The variables of type  $\epsilon$  in the specifying axioms given by Specifications 1–9 thus range over both standard and nonstandard constructions in a normal general model with nonstandard constructions. We will examine some basic results about having nonstandard constructions present in a normal general model.

**Lemma 6.3.2** *Let  $\mathcal{M}$  be a normal general model for  $\mathcal{Q}_0^{\text{uqe}}$  and  $\varphi \in \text{assign}(\mathcal{M})$ . Suppose  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\epsilon)$  is a nonstandard construction. Then  $\mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \mathbf{A}_\epsilon \rrbracket_\gamma) = \mathbf{F}$  if  $\gamma = o$  and  $\mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \mathbf{A}_\epsilon \rrbracket_\gamma)$  is undefined if  $\gamma \neq o$ .*

**Proof** Immediate from the semantics of evaluation.  $\square$

**Lemma 6.3.3** *Let  $\mathcal{M}$  be a normal general model for  $\mathcal{Q}_0^{\text{uqe}}$  and  $\varphi \in \text{assign}(\mathcal{M})$ .*

1. *If  $\mathcal{V}_\varphi^{\mathcal{M}}(\text{app}_{\epsilon\epsilon\epsilon} \mathbf{x}_\epsilon \mathbf{y}_\epsilon)$  is defined, then  $\varphi(\mathbf{x}_\epsilon)$  and  $\varphi(\mathbf{y}_\epsilon)$  are standard constructions iff  $\mathcal{V}_\varphi^{\mathcal{M}}(\text{app}_{\epsilon\epsilon\epsilon} \mathbf{x}_\epsilon \mathbf{y}_\epsilon)$  is a standard construction.*
2. *If  $\mathcal{V}_\varphi^{\mathcal{M}}(\text{abs}_{\epsilon\epsilon\epsilon} \mathbf{x}_\epsilon \mathbf{y}_\epsilon)$  is defined, then  $\varphi(\mathbf{x}_\epsilon)$  and  $\varphi(\mathbf{y}_\epsilon)$  are standard constructions iff  $\mathcal{V}_\varphi^{\mathcal{M}}(\text{abs}_{\epsilon\epsilon\epsilon} \mathbf{x}_\epsilon \mathbf{y}_\epsilon)$  is a standard construction.*



3. If  $\mathcal{V}_\varphi^{\mathcal{M}}(\text{cond}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{x}_\epsilon \mathbf{y}_\epsilon \mathbf{z}_\epsilon)$  is defined, then  $\varphi(\mathbf{x}_\epsilon)$ ,  $\varphi(\mathbf{y}_\epsilon)$ , and  $\varphi(\mathbf{z}_\epsilon)$  are standard constructions iff  $\mathcal{V}_\varphi^{\mathcal{M}}(\text{app}_{\epsilon\epsilon\epsilon} \mathbf{x}_\epsilon \mathbf{y}_\epsilon \mathbf{z}_\epsilon)$  is a standard construction.
4.  $\varphi(\mathbf{x}_\epsilon)$  is a standard construction iff  $\mathcal{V}_\varphi^{\mathcal{M}}(\text{quot}_{\epsilon\epsilon} \mathbf{x}_\epsilon)$  is a standard construction.
5. If  $\mathcal{V}_\varphi^{\mathcal{M}}(\text{eval}_{\epsilon\epsilon\epsilon} \mathbf{x}_\epsilon \mathbf{y}_\epsilon)$  is defined, then  $\varphi(\mathbf{x}_\epsilon)$  and  $\varphi(\mathbf{y}_\epsilon)$  are standard constructions iff  $\mathcal{V}_\varphi^{\mathcal{M}}(\text{eval}_{\epsilon\epsilon\epsilon} \mathbf{x}_\epsilon \mathbf{y}_\epsilon)$  is a standard construction.

**Proof**

**Part 1** Let  $\mathcal{V}_\varphi^{\mathcal{M}}(\text{app}_{\epsilon\epsilon\epsilon} \mathbf{x}_\epsilon \mathbf{y}_\epsilon)$  be defined. Assume  $\varphi(\mathbf{x}_\epsilon)$  and  $\varphi(\mathbf{y}_\epsilon)$  are standard constructions. Then  $\varphi(\mathbf{x}_\epsilon) = \mathcal{E}(\mathbf{A}_{\alpha\beta})$  and  $\varphi(\mathbf{y}_\epsilon) = \mathcal{E}(\mathbf{B}_\beta)$  for some wffs  $\mathbf{A}_{\alpha\beta}$  and  $\mathbf{B}_\beta$  by Specifications 6.4 and 6.5. Hence, by the definition of  $\mathcal{E}$ ,

$$\begin{aligned} & \mathcal{V}_\varphi^{\mathcal{M}}(\text{app}_{\epsilon\epsilon\epsilon} \mathbf{x}_\epsilon \mathbf{y}_\epsilon) \\ &= \mathcal{V}_\varphi^{\mathcal{M}}(\text{app}_{\epsilon\epsilon\epsilon} \mathcal{E}(\mathbf{A}_{\alpha\beta}) \mathcal{E}(\mathbf{B}_\beta)) \\ &= \mathcal{V}_\varphi^{\mathcal{M}}(\mathcal{E}(\mathbf{A}_{\alpha\beta} \mathbf{B}_\beta)), \end{aligned}$$

which is clearly a standard construction.

Now assume  $\mathcal{V}_\varphi^{\mathcal{M}}(\text{app}_{\epsilon\epsilon\epsilon} \mathbf{x}_\epsilon \mathbf{y}_\epsilon)$  is a standard construction. Then, by Specifications 4.1–21 and Specifications 6.4 and 6.5,

$$\begin{aligned} & \mathcal{V}_\varphi^{\mathcal{M}}(\text{app}_{\epsilon\epsilon\epsilon})(\varphi(\mathbf{x}_\epsilon))(\varphi(\mathbf{y}_\epsilon)) \\ &= \mathcal{V}_\varphi^{\mathcal{M}}(\text{app}_{\epsilon\epsilon\epsilon} \mathbf{x}_\epsilon \mathbf{y}_\epsilon) \\ &= \mathcal{V}_\varphi^{\mathcal{M}}(\text{app}_{\epsilon\epsilon\epsilon} \mathcal{E}(\mathbf{A}_{\alpha\beta}) \mathcal{E}(\mathbf{B}_\beta)) \\ &= \mathcal{V}_\varphi^{\mathcal{M}}(\text{app}_{\epsilon\epsilon\epsilon})(\mathcal{E}(\mathbf{A}_{\alpha\beta}))(\mathcal{E}(\mathbf{B}_\beta)) \end{aligned}$$

for some wffs  $\mathbf{A}_{\alpha\beta}$  and  $\mathbf{B}_\beta$ . Hence  $\varphi(\mathbf{x}_\epsilon) = \mathcal{E}(\mathbf{A}_{\alpha\beta})$  and  $\varphi(\mathbf{y}_\epsilon) = \mathcal{E}(\mathbf{B}_\beta)$  by Specification 4.24 and are thus standard constructions.

**Parts 2–5** Similar to Part 1. □

Let  $\varphi \in \text{assign}(\mathcal{M})$ . Suppose  $\mathcal{V}_\varphi^{\mathcal{M}}(\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{x}_\epsilon \mathbf{y}_\epsilon \mathbf{z}_\epsilon)$  is a standard construction. Does this imply that  $\varphi(\mathbf{x}_\epsilon)$ ,  $\varphi(\mathbf{y}_\epsilon)$ , and  $\varphi(\mathbf{z}_\epsilon)$  are standard constructions? The answer is no: Let  $\varphi(\mathbf{x}_\epsilon) = \mathcal{E}(\mathbf{c}_\alpha)$  for some constant  $\mathbf{c}_\alpha$  and  $\varphi(\mathbf{y}_\epsilon) = \varphi(\mathbf{z}_\epsilon)$  be a nonstandard construction such that  $\mathcal{V}_\varphi^{\mathcal{M}}(\text{var}_{\alpha\epsilon}^\alpha \mathbf{y}_\epsilon) = \top$ . Then  $\mathcal{V}_\varphi^{\mathcal{M}}(\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{x}_\epsilon \mathbf{y}_\epsilon \mathbf{z}_\epsilon) = \mathcal{E}(\mathbf{c}_\alpha)$  by Specifications 3.1, 6.2, 8.2, and 9.1.

However, the following result does hold:

**Lemma 6.3.4** *Let  $\mathcal{M}$  be a normal general model for  $\mathcal{Q}_0^{\text{hqe}}$  and  $\varphi \in \text{assign}(\mathcal{M})$ . If  $\varphi(\mathbf{x}_\epsilon)$ ,  $\varphi(\mathbf{y}_\epsilon)$ , and  $\mathcal{V}_\varphi^{\mathcal{M}}(\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{x}_\epsilon \mathbf{y}_\epsilon \mathbf{z}_\epsilon)$  are standard constructions and  $\mathcal{V}_\varphi^{\mathcal{M}}(\text{eval-free}_{\alpha\epsilon} \mathbf{z}_\epsilon) = \top$ , then  $\varphi(\mathbf{z}_\epsilon)$  is a standard construction.*

**Proof** Let  $\mathcal{V}_\varphi^{\mathcal{M}}(\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{x}_\epsilon \mathbf{y}_\epsilon \mathbf{z}_\epsilon) = \mathcal{E}(\mathbf{A}_\alpha)$  for some wff  $\mathbf{A}_\alpha$ . Then the proof of the lemma is by induction on the size of  $\mathbf{A}_\alpha$ . □

## 6.4 Example: Infinite Dependency

Having specified the logical constant  $\text{var}_{o\epsilon}$  in this section, we are now ready to present the following simple, but very interesting example.

Let  $\mathcal{M} = \langle \{\mathcal{D}_\alpha \mid \alpha \in \mathcal{T}\}, \mathcal{J} \rangle$  be a normal general model for  $\mathcal{Q}_0^{\text{uqe}}$  with  $\mathcal{D}_\epsilon = \{\mathcal{E}(\mathbf{A}_\alpha) \mid \mathbf{A}_\alpha \text{ is a wff}\}$  and  $\varphi \in \text{assign}(\mathcal{M})$ . Let  $\mathbf{A}_o$  be the simple formula

$$\forall x_\epsilon [\text{var}_{o\epsilon}^o x_\epsilon \supset \llbracket x_\epsilon \rrbracket_o]$$

involving evaluation. If we forget about evaluation,  $\mathbf{A}_o$  looks like a semantically close formula — which is not the case! By the semantics of universal quantification  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_o) = \mathbf{T}$  iff  $\mathcal{V}_{\varphi[x_\epsilon \mapsto \mathcal{E}(\mathbf{B}_\alpha)]}^{\mathcal{M}}(\text{var}_{o\epsilon}^o x_\epsilon \supset \llbracket x_\epsilon \rrbracket_o) = \mathbf{T}$  for every wff  $\mathbf{B}_\alpha$ . If  $\mathbf{B}_\alpha$  is not a variable of type  $o$ , then  $\mathcal{V}_{\varphi[x_\epsilon \mapsto \mathcal{E}(\mathbf{B}_\alpha)]}^{\mathcal{M}}(\text{var}_{o\epsilon}^o x_\epsilon) = \mathbf{F}$ , and so  $\mathcal{V}_{\varphi[x_\epsilon \mapsto \mathcal{E}(\mathbf{B}_\alpha)]}^{\mathcal{M}}(\text{var}_{o\epsilon}^o x_\epsilon \supset \llbracket x_\epsilon \rrbracket_o) = \mathbf{T}$ . If  $\mathbf{B}_\alpha$  is a variable  $\mathbf{y}_o$ , then

$$\begin{aligned} & \mathcal{V}_{\varphi[x_\epsilon \mapsto \mathcal{E}(\mathbf{y}_o)]}^{\mathcal{M}}(\llbracket \text{var}_{o\epsilon}^o x_\epsilon \supset \llbracket x_\epsilon \rrbracket_o \rrbracket) \\ &= \mathcal{V}_{\varphi[x_\epsilon \mapsto \mathcal{E}(\mathbf{y}_o)]}^{\mathcal{M}}(\llbracket x_\epsilon \rrbracket_o) \\ &= \mathcal{V}_{\varphi[x_\epsilon \mapsto \mathcal{E}(\mathbf{y}_o)]}^{\mathcal{M}}(\mathcal{E}^{-1}(\mathcal{V}_{\varphi[x_\epsilon \mapsto \mathcal{E}(\mathbf{y}_o)]}^{\mathcal{M}}(x_\epsilon))) \\ &= \mathcal{V}_{\varphi[x_\epsilon \mapsto \mathcal{E}(\mathbf{y}_o)]}^{\mathcal{M}}(\mathcal{E}^{-1}(\mathcal{E}(\mathbf{y}_o))) \\ &= \mathcal{V}_{\varphi[x_\epsilon \mapsto \mathcal{E}(\mathbf{y}_o)]}^{\mathcal{M}}(\mathbf{y}_o) \\ &= \varphi(\mathbf{y}_o). \end{aligned}$$

Hence  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_o) = \mathbf{T}$  iff  $\varphi(\mathbf{y}_o) = \mathbf{T}$  for all variables  $\mathbf{y}_o$  of type  $o$ . Therefore, not only is  $\mathbf{A}_o$  not semantically closed, its value in  $\mathcal{M}$  depends on the values assigned to infinitely many variables. In contrast, the value of any evaluation-free wff depends on at most finitely many variables.

## 7 Substitution

Our next task is to construct a proof system  $\mathcal{P}^{\text{uqe}}$  for  $\mathcal{Q}_0^{\text{uqe}}$  based on the proof system of  $\mathcal{Q}_0^{\text{u}}$ . We need a mechanism for substituting a wff  $\mathbf{A}_\alpha$  for a free variable  $\mathbf{x}_\alpha$  in another wff  $\mathbf{B}_\alpha$  so that we can perform beta-reduction in  $\mathcal{P}^{\text{uqe}}$ . Beta-reduction is performed in the proof system of  $\mathcal{Q}_0$  in a purely syntactic way using the basic properties of lambda-notation stated as Axioms 4<sub>1</sub>–4<sub>5</sub> in [2]. Due to the Variable Problem discussed in section 1,  $\mathcal{P}^{\text{uqe}}$  requires a semantics-dependent form of substitution. There is no easy way of extending or modifying Axioms 4<sub>1</sub>–4<sub>5</sub> to cover all function abstractions that contain evaluations. Instead, we will utilize a form of explicit substitution [1]. We will also utilize as well the basic properties of lambda-notation that remain valid in  $\mathcal{Q}_0^{\text{uqe}}$ .

The law of beta-reduction for  $\mathcal{Q}_0^{\text{u}}$  is expressed as the schema

$$\mathbf{A}_\alpha \downarrow \supset \llbracket [\lambda \mathbf{x}_\alpha \mathbf{B}_\beta] \mathbf{A}_\alpha \simeq \mathbf{S}_{\mathbf{A}_\alpha}^{\mathbf{x}_\alpha} \mathbf{B}_\beta \rrbracket$$

where  $\mathbf{A}_\alpha$  is free for  $\mathbf{x}_\alpha$  in  $\mathbf{B}_\beta$  and  $\mathbf{S}_{\mathbf{A}_\alpha}^{\mathbf{x}_\alpha} \mathbf{B}_\beta$  is the result of substituting  $\mathbf{A}_\alpha$  for each free occurrence of  $\mathbf{x}_\alpha$  in  $\mathbf{B}_\beta$ .<sup>3</sup> The law of beta-reduction for  $\mathcal{Q}_0^{\text{uqe}}$  will be

<sup>3</sup>Andrews uses  $\mathfrak{S}$  (with a dot) instead of  $\mathbf{S}$  for substitution in [2].

expressed by the schema

$$[\mathbf{A}_\alpha \downarrow \wedge \text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner \urcorner = \ulcorner \mathbf{C}_\beta \urcorner] \supset [\lambda \mathbf{x}_\alpha \mathbf{B}_\beta] \mathbf{A}_\alpha \simeq \mathbf{C}_\beta$$

without the syntactic side condition that  $\mathbf{A}_\alpha$  is free for  $\mathbf{x}_\alpha$  in  $\mathbf{B}_\beta$  and with the result of the substitution expressed by the wff  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner \urcorner$ . The logical constant  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$  was specified in the previous section. We will prove in this section that the law of beta-reduction for  $\mathcal{Q}_0^{\text{uqe}}$  stated above — in which substitution is represented by  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$  — is valid in  $\mathcal{Q}_0^{\text{uqe}}$ .

## 7.1 Requirements for $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$

The specification of  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$  needs to satisfy the following requirements:

**Requirement 1** *When  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner \urcorner$  is defined, its value must represent the wff $_{\beta}$  that results from substituting  $\mathbf{A}_\alpha$  for each free occurrence of  $\mathbf{x}_\alpha$  in  $\mathbf{B}_\beta$ . More precisely, for any normal general model  $\mathcal{M}$  for  $\mathcal{Q}_0^{\text{uqe}}$ , if*

$$\mathcal{M} \models [\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner \urcorner] \downarrow,$$

then

$$\mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner \urcorner \rrbracket_{\beta}) \simeq \mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)]}^{\mathcal{M}}(\mathbf{B}_\beta)$$

must be true for all  $\varphi \in \text{assign}(\mathcal{M})$  such that  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)$  is defined. Satisfying this requirement is straightforward when  $\mathbf{A}_\alpha$  and  $\mathbf{B}_\beta$  are evaluation-free. Since the semantics of evaluation involves a double application of  $\mathcal{V}_\varphi^{\mathcal{M}}$ , the specification of  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner \urcorner$  must include a double substitution when  $\mathbf{B}_\beta$  is an evaluation.

**Requirement 2**  *$\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner \urcorner$  must be undefined when substitution would result in a variable capture. To avoid variable capture we need to check whether a variable does not occur freely in a wff. We have specified the logical constant  $\text{not-free-in}_{o\epsilon\epsilon}$  to do this.*

**Requirement 3** *When  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner \urcorner$  is defined, its value must represent an evaluation-free wff $_{\beta}$ . Otherwise  $\llbracket \text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner \urcorner \rrbracket_{\beta}$  will be undefined. We will “cleanse” any evaluations that remain after a substitution by effectively replacing each wff of the form  $\ulcorner \llbracket \mathbf{A}_\epsilon \rrbracket_{\alpha} \urcorner$  with*

$$[\text{if} [\text{eval-free}_{o\epsilon}^{\alpha} \mathbf{A}_\epsilon] \mathbf{A}_\epsilon \perp_{\epsilon}].$$

We have specified the logical constant  $\text{cleanse}_{\epsilon\epsilon}$  to do this.

**Requirement 4** *When  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner \urcorner$  is defined, its value must be semantically closed. That is, the variables occurring in  $\mathbf{A}_\alpha$  or  $\mathbf{B}_\beta$  must not be allowed to escape outside of a quotation. To avoid such variable escape when a wff of the form  $\ulcorner \llbracket \mathbf{A}_\epsilon \rrbracket_{\alpha} \urcorner$  is cleansed as noted above, we need to enforce that  $\mathbf{A}_\epsilon$  is semantically closed. We have used the defined constant  $\text{syn-closed}_{o\epsilon}$  to do this.*

**Requirement 5**  $\text{sub}_{\epsilon\epsilon\epsilon} \lceil \mathbf{A}_\alpha \rceil \lceil \mathbf{x}_\alpha \rceil \lceil \mathbf{B}_\beta \rceil$  is defined in the cases corresponding to when substitution is defined in  $\mathcal{Q}_0$ . More precisely,  $\text{sub}_{\epsilon\epsilon\epsilon} \lceil \mathbf{A}_\alpha \rceil \lceil \mathbf{x}_\alpha \rceil \lceil \mathbf{B}_\beta \rceil$  is defined whenever  $\mathbf{A}_\alpha$  and  $\mathbf{B}_\beta$  are evaluation-free,  $\mathbf{A}_\alpha$  is defined, and substituting  $\mathbf{A}_\alpha$  for each free occurrence of  $\mathbf{x}_\alpha$  in  $\mathbf{B}_\beta$  does not result in a variable capture.

We will prove a series of lemmas that show (1) the properties that  $\text{not-free-in}_{o\epsilon\epsilon}$ ,  $\text{cleanse}_{\epsilon\epsilon}$ , and  $\text{sub}_{\epsilon\epsilon\epsilon}$  have and (2) that  $\text{sub}_{\epsilon\epsilon\epsilon}$  satisfies Requirements 1–5.

## 7.2 Evaluation-Free Wffs

**Proposition 7.2.1 (Meaning of  $\text{eval-free}_{o\epsilon}^\alpha$ )** Let  $\mathcal{M}$  be a normal general model for  $\mathcal{Q}_0^{\text{uqe}}$ .  $\mathcal{M} \models \text{eval-free}_{o\epsilon}^\alpha \lceil \mathbf{A}_\alpha \rceil$  iff  $\mathbf{A}_\alpha$  is evaluation-free.

**Proof** Immediate from the specification of  $\text{eval-free}_{o\epsilon}^\alpha$ .  $\square$

**Lemma 7.2.2 (Evaluation-Free)** Let  $\mathcal{M}$  be a normal general model for  $\mathcal{Q}_0^{\text{uqe}}$  and  $\mathbf{A}_\alpha$  and  $\mathbf{B}_\beta$  be evaluation-free.

1.  $\text{not-free-in}_{o\epsilon\epsilon} \lceil \mathbf{x}_\alpha \rceil \lceil \mathbf{B}_\beta \rceil$ ,  $\text{syn-closed}_{o\epsilon} \lceil \mathbf{A}_\alpha \rceil$ ,  $\text{cleanse}_{\epsilon\epsilon} \lceil \mathbf{B}_\beta \rceil$ , and  $\text{sub}_{\epsilon\epsilon\epsilon} \lceil \mathbf{A}_\alpha \rceil \lceil \mathbf{x}_\alpha \rceil \lceil \mathbf{B}_\beta \rceil$  are invariable.
2. If  $\mathcal{M} \models \text{not-free-in}_{o\epsilon\epsilon} \lceil \mathbf{x}_\alpha \rceil \lceil \mathbf{B}_\beta \rceil$ , then  $\mathbf{B}_\beta$  is independent of  $\{\mathbf{x}_\alpha\}$  in  $\mathcal{M}$ .
3. If  $\mathcal{M} \models \text{not-free-in}_{o\epsilon\epsilon} \lceil \mathbf{x}_\alpha \rceil \lceil \mathbf{B}_\beta \rceil$ , then  $\mathcal{M} \models \text{sub}_{\epsilon\epsilon\epsilon} \lceil \mathbf{A}_\alpha \rceil \lceil \mathbf{x}_\alpha \rceil \lceil \mathbf{B}_\beta \rceil = \lceil \mathbf{B}_\beta \rceil$ .
4.  $\mathcal{M} \models \text{cleanse}_{\epsilon\epsilon} \lceil \mathbf{B}_\beta \rceil = \lceil \mathbf{B}_\beta \rceil$ .
5. Either  $\mathcal{M} \models \text{sub}_{\epsilon\epsilon\epsilon} \lceil \mathbf{A}_\alpha \rceil \lceil \mathbf{x}_\alpha \rceil \lceil \mathbf{B}_\beta \rceil = \lceil \mathbf{C}_\beta \rceil$  for some evaluation-free  $\mathbf{C}_\beta$  or  $\mathcal{M} \models [\text{sub}_{\epsilon\epsilon\epsilon} \lceil \mathbf{A}_\alpha \rceil \lceil \mathbf{x}_\alpha \rceil \lceil \mathbf{B}_\beta \rceil] \uparrow$ .
6.  $\mathcal{M} \models \sim[\text{not-free-in}_{o\epsilon\epsilon} \lceil \mathbf{x}_\alpha \rceil \lceil \mathbf{B}_\beta \rceil]$  for at most finitely many variables  $\mathbf{x}_\alpha$ .

**Proof** Parts 1–5 follow straightforwardly by induction on the size of  $\lceil \mathbf{B}_\beta \rceil$ . Part 6 follows from the fact that  $\mathcal{M} \models \sim[\text{not-free-in}_{o\epsilon\epsilon} \lceil \mathbf{x}_\alpha \rceil \lceil \mathbf{B}_\beta \rceil]$  implies  $\lceil \mathbf{x}_\alpha \rceil$  occurs in  $\lceil \mathbf{B}_\beta \rceil$ .  $\square$

By virtue of Lemma 7.2.2 (particularly part 1), several standard definitions of predicate logic that are not applicable to wffs in general are applicable to evaluation-free wffs. Let  $\mathbf{A}_\alpha$ ,  $\mathbf{B}_\beta$ , and  $\mathbf{C}_o$  be evaluation-free wffs. A variable  $\mathbf{x}_\alpha$  is *bound in  $\mathbf{B}_\beta$*  if  $\text{not-free-in}_{o\epsilon\epsilon} \lceil \mathbf{x}_\alpha \rceil \lceil \mathbf{B}_\beta \rceil$  denotes  $\top$  and is *free in  $\mathbf{B}_\beta$*  if  $\text{not-free-in}_{o\epsilon\epsilon} \lceil \mathbf{x}_\alpha \rceil \lceil \mathbf{B}_\beta \rceil$  denotes  $\text{F}$ .  $\mathbf{A}_\alpha$  is *syntactically closed* if  $\text{syn-closed}_{o\epsilon} \lceil \mathbf{A}_\alpha \rceil$  denotes  $\top$ . A *universal closure* of  $\mathbf{C}_o$  is a formula

$$\forall \mathbf{x}_{\alpha_1}^1 \cdots \forall \mathbf{x}_{\alpha_n}^n \mathbf{C}_o$$

such that  $\mathbf{y}_\beta$  is free in  $\mathbf{C}_o$  iff  $\mathbf{y}_\beta \in \{\mathbf{x}_{\alpha_1}^1, \dots, \mathbf{x}_{\alpha_n}^n\}$ .

**Lemma 7.2.3 (Universal Closures)** *Let  $\mathcal{M}$  be a normal general model for  $\mathcal{Q}_0^{\text{uqe}}$ ,  $\mathbf{A}_o$  be an evaluation-free formula, and  $\mathbf{B}_o$  be a universal closure of  $\mathbf{A}_o$ .*

1.  $\mathbf{B}_o$  is syntactically closed.
2.  $\mathcal{M} \models \mathbf{A}_o$  iff  $\mathcal{M} \models \mathbf{B}_o$ .

**Proof** Part 1 follows from the definitions of universal closure and syntactically closed. Part 2 follows from the semantics of universal quantification.  $\square$

**Note 18 (Syntactically Closed)** It is clearly decidable whether an evaluation-free wff is syntactically closed. Is it also decidable whether a non-evaluation-free wff  $\mathbf{A}_\alpha$  is syntactically closed (i.e.,  $\models \text{syn-closed}_{o\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner$  holds)? Since  $\mathcal{Q}_0^{\text{uqe}}$  is undecidable, it follows that it is undecidable whether  $\models \text{syn-closed}_{o\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner$  holds when  $\mathbf{A}_\alpha$  has the form

$$\llbracket \text{if } \mathbf{B}_o \ulcorner \mathbf{c}_\alpha \urcorner \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \rrbracket_\alpha,$$

where  $\mathbf{c}_\alpha$  is a primitive constant. Therefore, it is undecidable whether a non-evaluation-free wff is syntactically closed.

**Lemma 7.2.4 (Semantically Closed)** *Let  $\mathcal{M}$  be a normal general model for  $\mathcal{Q}_0^{\text{uqe}}$ .*

1. If  $\mathbf{A}_\alpha$  is evaluation-free and syntactically closed, then  $\mathbf{A}_\alpha$  is semantically closed.
2. If  $\mathbf{A}_\epsilon$  is semantically closed, then either  $\mathcal{M} \models \mathbf{A}_\epsilon = \ulcorner \mathbf{B}_\beta \urcorner$  for some  $\mathbf{B}_\beta$  or  $\mathcal{M} \models \llbracket \mathbf{A}_\epsilon \rrbracket_\gamma \simeq \perp_\gamma$  for all  $\gamma \in \mathcal{T}$ .
3. If  $\mathbf{A}_\epsilon$  is semantically closed,  $\mathcal{M} \models \text{syn-closed}_{o\epsilon} \mathbf{A}_\epsilon$ , and  $\mathcal{M} \models \text{eval-free}_{o\epsilon}^\alpha \mathbf{A}_\epsilon$ , then  $\llbracket \mathbf{A}_\epsilon \rrbracket_\alpha$  is semantically closed.

**Proof**

**Part 1** Follows immediately from part 2 of Lemma 7.2.2.

**Part 2** Assume  $\mathbf{A}_\epsilon$  is semantically closed. Let  $\varphi \in \text{assign}(\mathcal{M})$ . If  $V_\varphi^{\mathcal{M}}(\mathbf{A}_\epsilon)$  is undefined or  $\mathcal{E}^{-1}(V_\varphi^{\mathcal{M}}(\mathbf{A}_\epsilon))$  is undefined, then  $\mathcal{M} \models \llbracket \mathbf{A}_\epsilon \rrbracket_\gamma \simeq \perp_\gamma$  for all  $\gamma \in \mathcal{T}$ . So we may assume  $\mathcal{E}^{-1}(V_\varphi^{\mathcal{M}}(\mathbf{A}_\epsilon))$  is some wff  $\mathbf{B}_\beta$ . Then  $V_\varphi^{\mathcal{M}}(\ulcorner \mathbf{B}_\beta \urcorner) = \mathcal{E}(\mathbf{B}_\beta) = \mathcal{E}(\mathcal{E}^{-1}(V_\varphi^{\mathcal{M}}(\mathbf{A}_\epsilon))) = V_\varphi^{\mathcal{M}}(\mathbf{A}_\epsilon)$ . The hypothesis implies  $\mathcal{E}^{-1}(V_\varphi^{\mathcal{M}}(\mathbf{A}_\epsilon))$  does not depend on  $\varphi$ . Hence  $\mathcal{M} \models \mathbf{A}_\epsilon = \ulcorner \mathbf{B}_\beta \urcorner$ .

**Part 3** Assume (a)  $\mathbf{A}_\epsilon$  is semantically closed, (b)  $\mathcal{M} \models \text{syn-closed}_{o\epsilon} \mathbf{A}_\epsilon$ , and (c)  $\mathcal{M} \models \text{eval-free}_{o\epsilon}^\alpha \mathbf{A}_\epsilon$ . (a) and part 2 of this lemma imply either there is some  $\mathbf{B}_\alpha$  such that (d)  $\mathcal{M} \models \mathbf{A}_\epsilon = \ulcorner \mathbf{B}_\alpha \urcorner$  or  $\mathcal{M} \models \llbracket \mathbf{A}_\epsilon \rrbracket_\alpha \simeq \perp_\alpha$ .  $\perp_\alpha$  is semantically closed, so we may assume (d). (b), (c), and (d) imply (e)  $\mathcal{M} \models \text{syn-closed}_{o\epsilon} \ulcorner \mathbf{B}_\alpha \urcorner$  and (f)  $\mathcal{M} \models \text{eval-free}_{o\epsilon} \ulcorner \mathbf{B}_\alpha \urcorner$ . (f) implies  $\mathbf{B}_\alpha$  is evaluation-free by Proposition 7.2.1, and this and (e) imply  $\mathbf{B}_\alpha$  is syntactically closed by part 1 of Lemma 7.2.2. Thus  $\mathbf{B}_\alpha$  is semantically closed by part 1 of this lemma. Therefore,  $\llbracket \mathbf{A}_\epsilon \rrbracket_\alpha$  is semantically closed since  $\mathcal{M} \models \mathbf{B}_\alpha \simeq \llbracket \ulcorner \mathbf{B}_\alpha \urcorner \rrbracket_\alpha$  by (f) and  $\mathcal{M} \models \llbracket \ulcorner \mathbf{B}_\alpha \urcorner \rrbracket_\alpha = \llbracket \mathbf{A}_\epsilon \rrbracket_\alpha$  by (d).  $\square$

### 7.3 Properties of not-free-in<sub>oεε</sub>

**Lemma 7.3.1 (Not Free In)** *Let  $\mathcal{M}$  be a normal general model for  $\mathcal{Q}_0^{\text{uqe}}$ .*

1. *If  $X$  is a set of variables such that  $\mathcal{M} \models \text{not-free-in}_{oεε} \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner$  for all  $\mathbf{x}_\alpha \in X$ , then  $\mathbf{B}_\beta$  is independent of  $X$  in  $\mathcal{M}$ .*
2. *If  $\mathcal{M} \models \text{not-free-in}_{oεε} \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner$ , then*

$$V_\varphi^{\mathcal{M}}(\mathbf{B}_\beta) \simeq V_{\varphi[\mathbf{x}_\alpha \mapsto d]}^{\mathcal{M}}(\mathbf{B}_\beta)$$

*for all  $\varphi \in \text{assign}(\mathcal{M})$  and all  $d \in \mathcal{D}_\alpha$ .*

#### Proof

**Part 1** Let  $X$  be a set of variables. Without loss of generality, we may assume that  $X$  is nonempty. We will show that, if

$$\mathcal{M} \models \text{not-free-in}_{oεε} \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{D}_\delta \urcorner \text{ for all } \mathbf{x}_\alpha \in X \text{ [designated } H(\ulcorner \mathbf{D}_\delta \urcorner, X)\text{]},$$

then

$$\mathbf{D}_\delta \text{ is independent of } X \text{ in } \mathcal{M} \text{ [designated } C(\mathbf{D}_\delta, X)\text{]}.$$

Our proof is by induction on the complexity of  $\mathbf{D}_\delta$ . There are 9 cases corresponding to the 9 parts of Specification 7 used to specify  $\text{not-free-in}_{oεε} \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{D}_\delta \urcorner$ .

**Case 1:**  $\mathbf{D}_\delta$  is a variable  $\mathbf{x}_\alpha$ . Assume  $H(\ulcorner \mathbf{x}_\alpha \urcorner, X)$  is true. Then  $\mathbf{x}_\alpha \notin X$  by the specification of  $\text{not-free-in}_{oεε}$ . Hence  $C(\mathbf{x}_\alpha, X)$  is obviously true.

**Case 2:**  $\mathbf{D}_\delta$  is a primitive constant  $\mathbf{c}_\alpha$ . Then  $C(\mathbf{c}_\alpha, X)$  is true since every primitive constant is semantically closed by Proposition 3.3.3.

**Case 3:**  $\mathbf{D}_\delta$  is  $\mathbf{A}_{\alpha\beta}\mathbf{B}_\beta$ . Assume  $H(\ulcorner \mathbf{A}_{\alpha\beta}\mathbf{B}_\beta \urcorner, X)$  is true. Then  $H(\ulcorner \mathbf{A}_{\alpha\beta} \urcorner, X)$  and  $H(\ulcorner \mathbf{B}_\beta \urcorner, X)$  are true by the specification of  $\text{not-free-in}_{oεε}$ . Hence  $C(\mathbf{A}_{\alpha\beta}, X)$  and  $C(\mathbf{B}_\beta, X)$  are true by the induction hypothesis. These imply  $C(\mathbf{A}_{\alpha\beta}\mathbf{B}_\beta, X)$  by the semantics of function application.

**Case 4:**  $\mathbf{D}_\delta$  is  $\lambda\mathbf{x}_\alpha\mathbf{A}_\beta$ . Assume  $H(\ulcorner \lambda\mathbf{x}_\alpha\mathbf{A}_\beta \urcorner, X)$  is true.  $C(\lambda\mathbf{x}_\alpha\mathbf{A}_\beta, \{\mathbf{x}_\alpha\})$  is true by the semantics of function abstraction.  $H(\ulcorner \lambda\mathbf{x}_\alpha\mathbf{A}_\beta \urcorner, X)$  implies  $H(\ulcorner \mathbf{A}_\beta \urcorner, X \setminus \{\mathbf{x}_\alpha\})$  by the specification of  $\text{not-free-in}_{oεε}$ . Hence  $C(\mathbf{A}_\beta, X \setminus \{\mathbf{x}_\alpha\})$  is true by the induction hypothesis. This implies  $C(\lambda\mathbf{x}_\alpha\mathbf{A}_\beta, X \setminus \{\mathbf{x}_\alpha\})$  by the semantics of function abstraction. Therefore,  $C(\lambda\mathbf{x}_\alpha\mathbf{A}_\beta, X)$  holds.

**Case 5:**  $\mathbf{D}_\delta$  is if  $\mathbf{A}_o\mathbf{B}_\alpha\mathbf{C}_\alpha$ . Similar to Case 3.

**Case 6:**  $\mathbf{D}_\delta$  is  $\ulcorner \mathbf{A}_\alpha \urcorner$ . Then  $C(\ulcorner \mathbf{A}_\alpha \urcorner, X)$  is true since every quotation is semantically closed by Proposition 3.3.3.

**Case 7:**  $D_\delta$  is  $\llbracket \mathbf{A}_\epsilon \rrbracket_\alpha$ . Assume  $H(\ulcorner \llbracket \mathbf{A}_\epsilon \rrbracket_\alpha \urcorner, X)$  is true. Then (a)  $\mathcal{M} \models \text{syn-closed}_{o_\epsilon} \ulcorner \mathbf{A}_\epsilon \urcorner$ , (b)  $\mathcal{M} \models \text{eval-free}_{o_\epsilon}^\epsilon \ulcorner \mathbf{A}_\epsilon \urcorner$ , (c)  $\mathcal{M} \models \text{eval-free}_{o_\epsilon}^\alpha \mathbf{A}_\epsilon$ , and (d)  $H(\mathbf{A}_\epsilon, X)$  by the specification of  $\text{not-free-in}_{o_\epsilon\epsilon}$  and the fact  $X$  is nonempty. (a) and (b) imply (e)  $\mathbf{A}_\epsilon$  is semantically closed by Proposition 7.2.1 and part 1 of Lemma 7.2.4. (e) and part 2 of Lemma 7.2.4 implies either  $\llbracket \mathbf{A}_\epsilon \rrbracket_\alpha$  is semantically closed or (f)  $\mathcal{E}^{-1}(\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\epsilon))$  is defined for all  $\varphi \in \text{assign}(\mathcal{M})$ . So we may assume (f). (c) and (f) imply (g)  $\mathcal{E}^{-1}(\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\epsilon))$  is an evaluation-free wff $_\alpha$  for all  $\varphi \in \text{assign}(\mathcal{M})$ , and thus the complexity of  $\mathcal{E}^{-1}(\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\epsilon))$  is less than the complexity of  $\llbracket \mathbf{A}_\epsilon \rrbracket_\alpha$  (for any  $\varphi \in \text{assign}(\mathcal{M})$ ). Hence (d) implies  $C(\mathcal{E}^{-1}(\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\epsilon)), X)$  by the induction hypothesis. Let  $\varphi, \varphi' \in \text{assign}(\mathcal{M})$  such that  $\varphi(\mathbf{x}_\alpha) = \varphi'(\mathbf{x}_\alpha)$  whenever  $\mathbf{x}_\alpha \notin X$ . Then

$$\mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \mathbf{A}_\epsilon \rrbracket_\beta) \tag{1}$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\mathcal{E}^{-1}(\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\epsilon))) \tag{2}$$

$$\simeq \mathcal{V}_{\varphi'}^{\mathcal{M}}(\mathcal{E}^{-1}(\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\epsilon))) \tag{3}$$

$$\simeq \mathcal{V}_{\varphi'}^{\mathcal{M}}(\mathcal{E}^{-1}(\mathcal{V}_{\varphi'}^{\mathcal{M}}(\mathbf{A}_\epsilon))) \tag{4}$$

$$\simeq \mathcal{V}_{\varphi'}^{\mathcal{M}}(\llbracket \mathbf{A}_\epsilon \rrbracket_\beta). \tag{5}$$

(2) is by (g) and the semantics of evaluation; (3) is by  $C(\mathcal{E}^{-1}(\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\epsilon)), X)$ ; (4) is by (e); and (5) is again by (g) and the semantics of evaluation. This implies  $C(\llbracket \mathbf{A}_\epsilon \rrbracket_\beta, X)$ .

**Part 2** This part of the lemma is the special case of part 1 when  $X$  is a singleton.  $\square$

## 7.4 Properties of $\text{cleanse}_{\epsilon\epsilon}$

**Lemma 7.4.1 (Cleanse)** *Let  $\mathcal{M}$  be a normal general model for  $\mathcal{Q}_0^{\text{uqe}}$ .*

1. *If  $\mathcal{M} \models [\text{cleanse}_{\epsilon\epsilon} \ulcorner D_\delta \urcorner]_\downarrow$ , then  $\text{cleanse}_{\epsilon\epsilon} \ulcorner D_\delta \urcorner$  is semantically closed and*

$$\mathcal{M} \models \text{eval-free}_{o_\epsilon}^\delta [\text{cleanse}_{\epsilon\epsilon} \ulcorner D_\delta \urcorner].$$

2. *Either  $\mathcal{M} \models \text{cleanse}_{\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner = \ulcorner \mathbf{B}_\alpha \urcorner$  for some evaluation-free  $\mathbf{B}_\alpha$  or  $\mathcal{M} \models \llbracket \text{cleanse}_{\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \rrbracket_\gamma \simeq \perp_\gamma$  for all  $\gamma \in \mathcal{T}$ .*

3. *If  $\mathbf{C}_\gamma$  contains an evaluation  $\llbracket \mathbf{A}_\epsilon \rrbracket_\alpha$  not in a quotation such that, for some variable  $\mathbf{x}_\beta$ ,  $\mathcal{M} \models \sim[\text{not-free-in}_{o_\epsilon\epsilon} \ulcorner \mathbf{x}_\beta \urcorner \ulcorner \mathbf{A}_\epsilon \urcorner]$ , then*

$$\mathcal{M} \models [\text{cleanse}_{\epsilon\epsilon} \ulcorner \mathbf{C}_\gamma \urcorner]_\uparrow.$$

4. *If  $\mathcal{M} \models [\text{cleanse}_{\epsilon\epsilon} \ulcorner D_\delta \urcorner]_\downarrow$ , then*

$$\mathcal{M} \models \llbracket \text{cleanse}_{\epsilon\epsilon} \ulcorner D_\delta \urcorner \rrbracket_\delta \simeq D_\delta$$

**Proof** Let  $A(\ulcorner \mathbf{D}_\delta \urcorner)$  mean  $\text{cleanse}(\ulcorner \mathbf{D}_\delta \urcorner)$ .

**Part 1** Our proof is by induction on the complexity of  $\mathbf{D}_\delta$ . There are 7 cases corresponding to the 7 parts of Specification 8 used to specify  $A(\ulcorner \mathbf{D}_\delta \urcorner)$ .

**Cases 1, 2, and 6:**  $\mathbf{D}_\delta$  is a variable, primitive constant, or quotation. Then  $\mathcal{M} \models A(\ulcorner \mathbf{D}_\delta \urcorner) = \ulcorner \mathbf{D}_\delta \urcorner$  by the specification of  $\text{cleanse}_{\epsilon\epsilon}$ . Hence  $A(\ulcorner \mathbf{D}_\delta \urcorner)$  is semantically closed since a quotation is semantically closed by Proposition 3.3.3 and  $\mathcal{M} \models \text{eval-free}_{\rho\epsilon}^\delta A(\ulcorner \mathbf{D}_\delta \urcorner)$  since a variable, primitive constant, or quotation is evaluation-free.

**Case 3:**  $\mathbf{D}_\delta$  is  $\mathbf{A}_{\alpha\beta}\mathbf{B}_\beta$ . Assume  $\mathcal{M} \models A(\ulcorner \mathbf{A}_{\alpha\beta}\mathbf{B}_\beta \urcorner) \downarrow$ . Then  $\mathcal{M} \models A(\ulcorner \mathbf{A}_{\alpha\beta} \urcorner) \downarrow$  and  $\mathcal{M} \models A(\ulcorner \mathbf{B}_\beta \urcorner) \downarrow$  by the specification of  $\text{cleanse}_{\epsilon\epsilon}$ . It follows that  $A(\ulcorner \mathbf{A}_{\alpha\beta}\mathbf{B}_\beta \urcorner)$  is semantically closed and  $\mathcal{M} \models \text{eval-free}_{\rho\epsilon}^\alpha A(\ulcorner \mathbf{A}_{\alpha\beta}\mathbf{B}_\beta \urcorner)$  by the induction hypothesis and the specification of  $\text{cleanse}_{\epsilon\epsilon}$ .

**Case 4:**  $\mathbf{D}_\delta$  is  $\lambda \mathbf{x}_\beta \mathbf{A}_\alpha$ . Similar to Case 3.

**Case 5:**  $\mathbf{D}_\delta$  is  $\text{if } \mathbf{A}_\alpha \mathbf{B}_\alpha \mathbf{C}_\alpha$ . Similar to the proof of Case 3.

**Case 7:**  $\mathbf{D}_\delta$  is  $\llbracket \mathbf{A}_\epsilon \rrbracket_\alpha$ . Assume (a)  $\mathcal{M} \models A(\ulcorner \llbracket \mathbf{A}_\epsilon \rrbracket_\alpha \urcorner) \downarrow$ . (a) implies (b)  $\mathcal{M} \models \text{syn-closed}_{\rho\epsilon} A(\ulcorner \mathbf{A}_\epsilon \urcorner)$ , (c)  $\mathcal{M} \models \text{eval-free}_{\rho\epsilon}^\alpha \llbracket A(\ulcorner \mathbf{A}_\epsilon \urcorner) \rrbracket_\epsilon$ , and

$$(d) \mathcal{M} \models A(\ulcorner \llbracket \mathbf{A}_\epsilon \rrbracket_\alpha \urcorner) \simeq \llbracket A(\ulcorner \mathbf{A}_\epsilon \urcorner) \rrbracket_\epsilon$$

by the specification of  $\text{cleanse}_{\epsilon\epsilon}$ . (a) implies (e)  $\mathcal{M} \models A(\ulcorner \mathbf{A}_\epsilon \urcorner) \downarrow$ , and (e) implies (f)  $A(\ulcorner \mathbf{A}_\epsilon \urcorner)$  is semantically closed and (g)  $\mathcal{M} \models \text{eval-free}_{\rho\epsilon}^\epsilon A(\ulcorner \mathbf{A}_\epsilon \urcorner)$  by the induction hypothesis. (b), (f), and (g) imply (h)  $\llbracket A(\ulcorner \mathbf{A}_\epsilon \urcorner) \rrbracket_\epsilon$  is semantically closed by part 3 of Lemma 7.2.4. Therefore,  $A(\ulcorner \llbracket \mathbf{A}_\epsilon \rrbracket_\alpha \urcorner)$  is semantically closed by (d) and (h).

**Part 2** Follows easily from part 1 of this lemma and part 2 of Lemma 7.2.4.

**Part 3** Follows immediately from the specification of  $\text{cleanse}_{\epsilon\epsilon}$ .

**Part 4** Assume

$$\mathcal{M} \models A(\ulcorner \mathbf{D}_\delta \urcorner) \downarrow \quad [\text{designated } H(\ulcorner \mathbf{D}_\delta \urcorner)].$$

We must show that

$$\mathcal{M} \models \llbracket A(\ulcorner \mathbf{D}_\delta \urcorner) \rrbracket_\delta \simeq \mathbf{D}_\delta \quad [\text{designated } C(\ulcorner \mathbf{D}_\delta \urcorner)].$$

Our proof is by induction on the complexity of  $\mathbf{D}_\delta$ . There are 7 cases corresponding to the 7 parts of Specification 8 used to specify  $A(\ulcorner \mathbf{D}_\delta \urcorner)$ . Let  $\varphi \in \text{assign}(\mathcal{M})$ .

**Case 1:**  $\mathbf{D}_\delta$  is  $\mathbf{x}_\alpha$ . Then

$$\mathcal{V}_\varphi^\mathcal{M}(\llbracket A(\ulcorner \mathbf{x}_\alpha \urcorner) \rrbracket_\alpha) \tag{1}$$

$$\simeq \mathcal{V}_\varphi^\mathcal{M}(\llbracket \ulcorner \mathbf{x}_\alpha \urcorner \rrbracket_\alpha) \tag{2}$$

$$\simeq \mathcal{V}_\varphi^\mathcal{M}(\mathbf{x}_\alpha). \tag{3}$$



(2) is by the specification of  $\text{cleanse}_{\epsilon\epsilon}$ , and (3) is by the fact that  $\mathbf{x}_\alpha$  is evaluation-free and the semantics of evaluation. Therefore,  $C(\ulcorner \mathbf{x}_\alpha \urcorner)$  holds.

**Case 2:**  $\mathbf{D}_\delta$  is a primitive constant  $\mathbf{c}_\alpha$ . Similar to Case 1.

**Case 3:**  $\mathbf{D}_\delta$  is  $\mathbf{A}_{\alpha\beta}\mathbf{B}_\beta$ .  $H(\ulcorner \mathbf{A}_{\alpha\beta}\mathbf{B}_\beta \urcorner)$  implies  $H(\ulcorner \mathbf{A}_{\alpha\beta} \urcorner)$  and  $H(\ulcorner \mathbf{B}_\beta \urcorner)$  by the specification of  $\text{cleanse}_{\epsilon\epsilon}$ . These imply  $C(\ulcorner \mathbf{A}_{\alpha\beta} \urcorner)$  and  $C(\ulcorner \mathbf{B}_\beta \urcorner)$  by the induction hypothesis. Then

$$\mathcal{V}_\varphi^{\mathcal{M}}(\llbracket A(\ulcorner \mathbf{A}_{\alpha\beta}\mathbf{B}_\beta \urcorner) \rrbracket_\alpha) \quad (1)$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \text{app}_{\epsilon\epsilon\epsilon} A(\ulcorner \mathbf{A}_{\alpha\beta} \urcorner) A(\ulcorner \mathbf{B}_\beta \urcorner) \rrbracket_\alpha) \quad (2)$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket A(\ulcorner \mathbf{A}_{\alpha\beta} \urcorner) \rrbracket_{\alpha\beta} \llbracket A(\ulcorner \mathbf{B}_\beta \urcorner) \rrbracket_\beta) \quad (3)$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_{\alpha\beta}\mathbf{B}_\beta). \quad (4)$$

(2) is by the specification of  $\text{cleanse}_{\epsilon\epsilon}$ ; (3) is by the semantics of  $\text{app}_{\epsilon\epsilon\epsilon}$  and evaluation; and (4) is by  $C(\ulcorner \mathbf{A}_{\alpha\beta} \urcorner)$  and  $C(\ulcorner \mathbf{B}_\beta \urcorner)$ . Therefore,  $C(\ulcorner \mathbf{A}_{\alpha\beta}\mathbf{B}_\beta \urcorner)$  holds.

**Case 4:**  $\mathbf{D}_\delta$  is  $\lambda\mathbf{x}_\beta\mathbf{A}_\alpha$ .  $H(\ulcorner \lambda\mathbf{x}_\beta\mathbf{A}_\alpha \urcorner)$  implies  $H(\ulcorner \mathbf{A}_\alpha \urcorner)$  by the specification of  $\text{cleanse}_{\epsilon\epsilon}$ . This implies  $C(\ulcorner \mathbf{A}_\alpha \urcorner)$  by the induction hypothesis and  $A(\ulcorner \mathbf{A}_\alpha \urcorner)$  is semantically closed by part 1 of this lemma. Then

$$\mathcal{V}_\varphi^{\mathcal{M}}(\llbracket A(\ulcorner \lambda\mathbf{x}_\beta\mathbf{A}_\alpha \urcorner) \rrbracket_{\alpha\beta}) \quad (1)$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \text{abs}_{\epsilon\epsilon\epsilon} \ulcorner \mathbf{x}_\beta \urcorner A(\ulcorner \mathbf{A}_\alpha \urcorner) \rrbracket_{\alpha\beta}) \quad (2)$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\lambda\mathbf{x}_\beta \llbracket A(\ulcorner \mathbf{A}_\alpha \urcorner) \rrbracket_\alpha) \quad (3)$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\lambda\mathbf{x}_\beta\mathbf{A}_\alpha). \quad (4)$$

(2) is by the specification of  $\text{cleanse}_{\epsilon\epsilon}$ ; (3) is by the semantics of  $\text{abs}_{\epsilon\epsilon\epsilon}$  and evaluation and the fact that  $A(\ulcorner \mathbf{A}_\alpha \urcorner)$  is semantically closed; and (4) is by  $C(\ulcorner \mathbf{A}_\alpha \urcorner)$ . Therefore,  $C(\ulcorner \lambda\mathbf{x}_\beta\mathbf{A}_\alpha \urcorner)$  holds.

**Case 5:**  $\mathbf{D}_\delta$  is if  $\mathbf{A}_o\mathbf{B}_\alpha\mathbf{C}_\alpha$ . Similar to Case 3.

**Case 6:**  $\mathbf{D}_\delta$  is  $\ulcorner \mathbf{A}_\alpha \urcorner$ . Similar to Case 1.

**Case 7:**  $\mathbf{D}_\delta$  is  $\llbracket \mathbf{A}_\epsilon \rrbracket_\alpha$ .  $H(\ulcorner \mathbf{A}_\epsilon \urcorner)$  is true by the proof for Case 7 of Part 1, and hence  $C(\ulcorner \mathbf{A}_\epsilon \urcorner)$  is true by the induction hypothesis. Then

$$\mathcal{V}_\varphi^{\mathcal{M}}(\llbracket A(\ulcorner \llbracket \mathbf{A}_\epsilon \rrbracket_\alpha \urcorner) \rrbracket_\alpha) \quad (1)$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \llbracket A(\ulcorner \mathbf{A}_\epsilon \urcorner) \rrbracket_\epsilon \rrbracket_\alpha) \quad (2)$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \mathbf{A}_\epsilon \rrbracket_\alpha). \quad (3)$$

(2) is by

$$\mathcal{M} \models A(\ulcorner \llbracket \mathbf{A}_\epsilon \rrbracket_\alpha \urcorner) \simeq \llbracket A(\ulcorner \mathbf{A}_\epsilon \urcorner) \rrbracket_\epsilon$$

shown in the proof for Case 7 of Part 1, and (3) is by  $C(\ulcorner \mathbf{A}_\epsilon \urcorner)$ . Therefore,  $C(\ulcorner \llbracket \mathbf{A}_\epsilon \rrbracket_\alpha \urcorner)$  holds.

□

## 7.5 Properties of $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$

**Lemma 7.5.1 (Substitution)** *Let  $\mathcal{M}$  be a normal general model for  $\mathcal{Q}_0^{\text{uqe}}$ .*

1. *If  $\mathcal{M} \models [\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner ] \downarrow$ , then  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner$  is semantically closed and*

$$\mathcal{M} \models \text{eval-free}_{o\epsilon}^\beta [\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner ].$$

2. *Either  $\mathcal{M} \models \text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner = \ulcorner \mathbf{C}_\beta \urcorner$  for some evaluation-free  $\mathbf{C}_\beta$  or  $\mathcal{M} \models \llbracket \text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner \rrbracket_\gamma \simeq \perp_\gamma$  for all  $\gamma \in \mathcal{T}$ .*
3. *If  $\mathcal{M} \models \text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\epsilon \urcorner = \ulcorner \mathbf{C}_\beta \urcorner$  for some  $\mathbf{C}_\beta$  and  $\mathcal{M} \models \text{eval-free}_{o\epsilon}^\beta \mathbf{B}_\epsilon$ , then  $\mathcal{M} \models \mathbf{B}_\epsilon = \ulcorner \mathbf{D}_\beta \urcorner$  for some evaluation-free  $\mathbf{D}_\beta$ .*
4. *If  $\mathbf{C}_\gamma$  contains an evaluation  $\llbracket \mathbf{B}_\epsilon \rrbracket_\beta$  not in a quotation such that, for some variable  $\mathbf{y}_\gamma$  with  $\mathbf{x}_\alpha \neq \mathbf{y}_\gamma$ ,*

$$\mathcal{M} \models \sim[\text{not-free-in}_{o\epsilon\epsilon} \ulcorner \mathbf{y}_\gamma \urcorner \llbracket \text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\epsilon \urcorner \rrbracket ],$$

then

$$\mathcal{M} \models [\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{C}_\gamma \urcorner ] \uparrow.$$

5. *If  $\mathcal{M} \models \text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{D}_\delta \urcorner = \ulcorner \mathbf{E}_\delta \urcorner$  for some  $\mathbf{E}_\delta$  and*

$$\mathcal{M} \models \text{not-free-in}_{o\epsilon\epsilon} \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{D}_\delta \urcorner,$$

then

$$\mathcal{M} \models \llbracket \text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{D}_\delta \urcorner \rrbracket_\delta \simeq \mathbf{D}_\delta.$$

6. *If  $\mathcal{M} \models \text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{D}_\delta \urcorner = \ulcorner \mathbf{E}_\delta \urcorner$  for some  $\mathbf{E}_\delta$ , then*

$$\mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{D}_\delta \urcorner \rrbracket_\delta) \simeq \mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)]}^{\mathcal{M}}(\mathbf{D}_\delta)$$

for all  $\varphi \in \text{assign}(\mathcal{M})$  such that  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)$  is defined.

7.  *$\mathcal{M} \models [\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner ] \downarrow$  whenever  $\mathbf{A}_\alpha$  and  $\mathbf{B}_\beta$  are evaluation-free,  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)$  is defined, and substituting  $\mathbf{A}_\alpha$  for each free occurrence of  $\mathbf{x}_\alpha$  in  $\mathbf{B}_\beta$  does not result in a variable capture.*

**Proof** Let  $S(\ulcorner \mathbf{D}_\delta \urcorner)$  mean  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{D}_\delta \urcorner$ .

**Part 1** Similar to the proof of part 1 of Lemma 7.4.1.

**Part 2** Follows easily from part 1 of this lemma and part 2 of Lemma 7.2.4.

**Part 3** Follows from Lemma 6.3.4.

**Part 4** Follows immediately from the specification of  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$ .

**Part 5** Assume

$$\mathcal{M} \models S(\ulcorner \mathbf{D}_\delta \urcorner) = \ulcorner \mathbf{E}_\delta \urcorner \text{ for some } \mathbf{E}_\delta \text{ [designated } H_1(\ulcorner \mathbf{D}_\delta \urcorner)]$$

and

$$\mathcal{M} \models \text{not-free-in}_{o\epsilon\epsilon} \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{D}_\delta \urcorner \text{ [designated } H_2(\ulcorner \mathbf{D}_\delta \urcorner)].$$

We must show that

$$\mathcal{M} \models \llbracket S(\ulcorner \mathbf{D}_\delta \urcorner) \rrbracket_\delta \simeq \mathbf{D}_\delta \text{ [designated } C(\ulcorner \mathbf{D}_\delta \urcorner)].$$

Our proof is by induction on the complexity of  $\mathbf{D}_\delta$ . There are 9 cases corresponding to the 9 parts of Specification 9 used to specify  $S(\ulcorner \mathbf{D}_\delta \urcorner)$ . Let  $\varphi \in \text{assign}(\mathcal{M})$ .

**Case 1:**  $\mathbf{D}_\delta$  is  $\mathbf{x}_\alpha$ . By the specification of  $\text{not-free-in}_{o\epsilon\epsilon}$ ,  $H_2(\ulcorner \mathbf{x}_\alpha \urcorner)$  does not hold in this case.

**Case 2:**  $\mathbf{D}_\delta$  is  $\mathbf{y}_\beta$  where  $\mathbf{x}_\alpha \neq \mathbf{y}_\beta$ . Then

$$\mathcal{V}_\varphi^{\mathcal{M}}(\llbracket S(\ulcorner \mathbf{y}_\beta \urcorner) \rrbracket_\beta) \tag{1}$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \ulcorner \mathbf{y}_\beta \urcorner \rrbracket_\beta) \tag{2}$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{y}_\beta). \tag{3}$$

(2) is by the specification of  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$ , and (3) is by semantics of evaluation and the fact that  $\mathbf{y}_\beta$  is evaluation-free. Therefore,  $C(\ulcorner \mathbf{y}_\beta \urcorner)$  holds.

**Case 3:**  $\mathbf{D}_\delta$  is a primitive constant  $\mathbf{c}_\beta$ . Similar to Case 2.

**Case 4:**  $\mathbf{D}_\delta$  is  $\mathbf{B}_{\beta\gamma}\mathbf{D}_\delta$ .  $H_1(\ulcorner \mathbf{B}_{\beta\gamma}\mathbf{D}_\delta \urcorner)$  implies  $H_1(\ulcorner \mathbf{B}_{\beta\gamma} \urcorner)$  and  $H_1(\ulcorner \mathbf{D}_\delta \urcorner)$  by the specification of  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$ .  $H_2(\ulcorner \mathbf{B}_{\beta\gamma}\mathbf{D}_\delta \urcorner)$  implies  $H_2(\ulcorner \mathbf{B}_{\beta\gamma} \urcorner)$  and  $H_2(\ulcorner \mathbf{D}_\delta \urcorner)$  by the specification of  $\text{not-free-in}_{o\epsilon\epsilon}$ . These imply  $C(\ulcorner \mathbf{B}_{\beta\gamma} \urcorner)$  and  $C(\ulcorner \mathbf{D}_\delta \urcorner)$  by the induction hypothesis. Then

$$\mathcal{V}_\varphi^{\mathcal{M}}(\llbracket S(\ulcorner \mathbf{B}_{\beta\gamma}\mathbf{D}_\delta \urcorner) \rrbracket_\beta) \tag{1}$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \text{app}_{\epsilon\epsilon\epsilon} S(\ulcorner \mathbf{B}_{\beta\gamma} \urcorner) S(\ulcorner \mathbf{D}_\delta \urcorner) \rrbracket_\alpha) \tag{2}$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket S(\ulcorner \mathbf{B}_{\beta\gamma} \urcorner) \rrbracket_{\beta\gamma} \llbracket S(\ulcorner \mathbf{D}_\delta \urcorner) \rrbracket_\gamma) \tag{3}$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_{\beta\gamma}\mathbf{D}_\delta). \tag{4}$$

(2) is by the specification of  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$ ; (3) is by the semantics of  $\text{app}_{\epsilon\epsilon\epsilon}$  and evaluation; and (4) is by  $C(\ulcorner \mathbf{B}_{\beta\gamma} \urcorner)$  and  $C(\ulcorner \mathbf{D}_\delta \urcorner)$ . Therefore,  $C(\ulcorner \mathbf{B}_{\beta\gamma}\mathbf{D}_\delta \urcorner)$  holds.

**Case 5:**  $\mathbf{D}_\delta$  is  $\lambda\mathbf{x}_\alpha\mathbf{B}_\beta$ .  $H_1(\ulcorner \lambda\mathbf{x}_\alpha\mathbf{B}_\beta \urcorner)$  implies  $\mathcal{M} \models [\text{cleanse}_{\epsilon\epsilon} \ulcorner \mathbf{B}_\beta \urcorner] \downarrow$  by the specification of  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$ . This implies that  $\text{cleanse}_{\epsilon\epsilon} \ulcorner \mathbf{B}_\beta \urcorner$  is semantically

closed by part 1 of Lemma 7.4.1. Then

$$\mathcal{V}_\varphi^M(\llbracket S(\lambda \mathbf{x}_\alpha \mathbf{B}_\beta) \rrbracket_{\beta\alpha}) \quad (1)$$

$$\simeq \mathcal{V}_\varphi^M(\llbracket \mathbf{abs}_{\epsilon\epsilon\epsilon} \ulcorner \mathbf{x}_\alpha \urcorner \mathbf{cleanse}_{\epsilon\epsilon} \ulcorner \mathbf{B}_\beta \urcorner \rrbracket_{\beta\alpha}) \quad (2)$$

$$\simeq \mathcal{V}_\varphi^M(\lambda \mathbf{x}_\alpha \llbracket \mathbf{cleanse}_{\epsilon\epsilon} \ulcorner \mathbf{B}_\beta \urcorner \rrbracket_\beta) \quad (3)$$

$$\simeq \mathcal{V}_\varphi^M(\lambda \mathbf{x}_\alpha \mathbf{B}_\beta). \quad (4)$$

(2) is by the specification of  $\mathbf{sub}_{\epsilon\epsilon\epsilon\epsilon}$ ; (3) is by the semantics of  $\mathbf{abs}_{\epsilon\epsilon\epsilon}$  and evaluation and the fact that  $\mathbf{cleanse}_{\epsilon\epsilon} \ulcorner \mathbf{B}_\beta \urcorner$  is semantically closed; and (4) is by part 4 of Lemma 7.4.1. Therefore,  $C(\ulcorner \lambda \mathbf{x}_\alpha \mathbf{B}_\beta \urcorner)$  holds.

**Case 6:**  $\mathbf{D}_\delta$  is  $\lambda \mathbf{y}_\beta \mathbf{B}_\gamma$  where  $\mathbf{x}_\alpha \neq \mathbf{y}_\beta$ .  $H_1(\ulcorner \lambda \mathbf{y}_\beta \mathbf{B}_\gamma \urcorner)$  implies

$$\mathcal{V}_\varphi^M(S(\ulcorner \lambda \mathbf{y}_\beta \mathbf{B}_\gamma \urcorner)) \simeq \mathcal{V}_\varphi^M(\mathbf{abs}_{\epsilon\epsilon\epsilon} \ulcorner \mathbf{y}_\beta \urcorner S(\ulcorner \mathbf{B}_\gamma \urcorner))$$

and  $H_1(\ulcorner \mathbf{B}_\gamma \urcorner)$  by the specification of  $\mathbf{sub}_{\epsilon\epsilon\epsilon\epsilon}$ .  $H_2(\ulcorner \lambda \mathbf{y}_\beta \mathbf{B}_\gamma \urcorner)$  implies  $H_2(\ulcorner \mathbf{B}_\gamma \urcorner)$  by the specification of  $\mathbf{not-free-in}_{o\epsilon\epsilon}$ . These imply  $C(\ulcorner \mathbf{B}_\gamma \urcorner)$  by the induction hypothesis and  $S(\ulcorner \mathbf{B}_\gamma \urcorner)$  is semantically closed by part 1 of this lemma. Then

$$\mathcal{V}_\varphi^M(\llbracket S(\lambda \mathbf{y}_\beta \mathbf{B}_\gamma) \rrbracket_{\gamma\beta}) \quad (1)$$

$$\simeq \mathcal{V}_\varphi^M(\llbracket \mathbf{abs}_{\epsilon\epsilon\epsilon} \ulcorner \mathbf{y}_\beta \urcorner S(\ulcorner \mathbf{B}_\gamma \urcorner) \rrbracket_{\gamma\beta}) \quad (2)$$

$$\simeq \mathcal{V}_\varphi^M(\lambda \mathbf{y}_\beta \llbracket S(\ulcorner \mathbf{B}_\gamma \urcorner) \rrbracket_\gamma) \quad (3)$$

$$\simeq \mathcal{V}_\varphi^M(\lambda \mathbf{y}_\beta \mathbf{B}_\gamma). \quad (4)$$

(2) is by the equation shown above; (3) is by the semantics of  $\mathbf{abs}_{\epsilon\epsilon\epsilon}$  and evaluation and the fact that  $S(\ulcorner \mathbf{B}_\gamma \urcorner)$  is semantically closed; and (4) is by  $C(\ulcorner \mathbf{B}_\gamma \urcorner)$ . Therefore,  $C(\ulcorner \lambda \mathbf{y}_\beta \mathbf{B}_\gamma \urcorner)$  holds.

**Case 7:**  $\mathbf{D}_\delta$  is if  $\mathbf{A}_o \mathbf{B}_\alpha \mathbf{C}_\alpha$ . Similar to Case 4.

**Case 8:**  $\mathbf{D}_\delta$  is  $\ulcorner \mathbf{B}_\beta \urcorner$ . Similar to Case 2.

**Case 9:**  $\mathbf{D}_\delta$  is  $\llbracket \mathbf{B}_\epsilon \rrbracket_\beta$ .  $H_1(\ulcorner \llbracket \mathbf{B}_\epsilon \rrbracket_{\beta\epsilon} \urcorner)$  implies

$$(a) \mathcal{M} \models \mathbf{eval-free}_{o\epsilon}^\beta \llbracket S(\ulcorner \mathbf{B}_\epsilon \urcorner) \rrbracket_\epsilon$$

and

$$(b) \mathcal{M} \models S(\ulcorner \llbracket \mathbf{B}_\epsilon \rrbracket_{\beta\epsilon} \urcorner) = S(\llbracket S(\ulcorner \mathbf{B}_\epsilon \urcorner) \rrbracket_\epsilon)$$

by the specification of  $\mathbf{sub}_{\epsilon\epsilon\epsilon\epsilon}$ . (a), (b), and  $H_1(\ulcorner \llbracket \mathbf{B}_\epsilon \rrbracket_{\beta\epsilon} \urcorner)$  imply  $H_1(\llbracket S(\ulcorner \mathbf{B}_\epsilon \urcorner) \rrbracket_\epsilon)$  by part 3 of this lemma, and so (c)  $\mathcal{M} \models \llbracket S(\ulcorner \mathbf{B}_\epsilon \urcorner) \rrbracket_\epsilon = \ulcorner \mathbf{C}_\beta \urcorner$  for some evaluation-free  $\mathbf{C}_\beta$ .  $H_1(\llbracket S(\ulcorner \mathbf{B}_\epsilon \urcorner) \rrbracket_\epsilon)$  implies  $H_1(\ulcorner \mathbf{B}_\epsilon \urcorner)$  by the specification of  $\mathbf{sub}_{\epsilon\epsilon\epsilon\epsilon}$ . By the specification of  $\mathbf{not-free-in}_{o\epsilon\epsilon}$ ,  $H_2(\ulcorner \llbracket \mathbf{B}_\epsilon \rrbracket_{\beta\epsilon} \urcorner)$  implies  $\mathcal{M} \models \mathbf{syn-closed}_{o\epsilon} \ulcorner \mathbf{B}_\epsilon \urcorner$  (hence  $H_2(\ulcorner \mathbf{B}_\epsilon \urcorner)$  by the definition of  $\mathbf{syn-closed}_{o\epsilon}$ ),  $\mathcal{M} \models \mathbf{eval-free}_{o\epsilon}^\epsilon \ulcorner \mathbf{B}_\epsilon \urcorner$ , and  $H_2(\llbracket \ulcorner \mathbf{B}_\epsilon \urcorner \rrbracket_\epsilon)$  (hence

$H_2(\mathbf{B}_\epsilon)$  by the semantics of evaluation).  $H_1(\ulcorner \mathbf{B}_\epsilon \urcorner)$  and  $H_2(\ulcorner \mathbf{B}_\epsilon \urcorner)$  imply  $C(\ulcorner \mathbf{B}_\epsilon \urcorner)$  by the induction hypothesis, and so (d)  $\mathcal{M} \models \llbracket S(\ulcorner \mathbf{B}_\epsilon \urcorner) \rrbracket_\epsilon \simeq \mathbf{B}_\epsilon$ . (c) and (d) imply (e)  $\mathcal{M} \models \mathbf{B}_\epsilon = \ulcorner \mathbf{C}_\beta \urcorner$ .  $H_1(\llbracket S(\ulcorner \mathbf{B}_\epsilon \urcorner) \rrbracket_\epsilon)$  and (c) imply  $H_1(\ulcorner \mathbf{C}_\beta \urcorner)$ .  $H_2(\mathbf{B}_\epsilon)$  and (e) implies  $H_2(\ulcorner \mathbf{C}_\beta \urcorner)$ .  $H_1(\ulcorner \mathbf{C}_\beta \urcorner)$  and  $H_2(\ulcorner \mathbf{C}_\beta \urcorner)$  imply  $C(\ulcorner \mathbf{C}_\beta \urcorner)$  by the inductive hypothesis. Then

$$\mathcal{V}_\varphi^{\mathcal{M}}(\llbracket S(\ulcorner \mathbf{B}_\epsilon \urcorner) \rrbracket_\beta) \quad (1)$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket S(\llbracket S(\ulcorner \mathbf{B}_\epsilon \urcorner) \rrbracket_\epsilon) \rrbracket_\beta) \quad (2)$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket S(\ulcorner \mathbf{C}_\beta \urcorner) \rrbracket_\beta) \quad (3)$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{C}_\beta) \quad (4)$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \ulcorner \mathbf{C}_\beta \urcorner \rrbracket_\beta) \quad (5)$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \mathbf{B}_\epsilon \rrbracket_\beta). \quad (6)$$

(2) is by (b); (3) is by (d) and (e); (4) is by  $C(\ulcorner \mathbf{C}_\beta \urcorner)$ ; (5) is by the semantics of evaluation and the fact  $\mathbf{C}_\beta$  is evaluation-free; and (6) is by (e). Therefore,  $C(\ulcorner \mathbf{B}_\epsilon \urcorner)$  holds.

#### Part 6 Assume

$$\mathcal{M} \models S(\ulcorner \mathbf{D}_\delta \urcorner) = \ulcorner \mathbf{E}_\delta \urcorner \text{ for some } \mathbf{E}_\delta \text{ [designated } H(\ulcorner \mathbf{D}_\delta \urcorner)]$$

We must show that

$$\mathcal{M} \models \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket S(\ulcorner \mathbf{D}_\delta \urcorner) \rrbracket_\delta) \simeq \mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)]}^{\mathcal{M}}(\mathbf{D}_\delta) \text{ for all } \varphi \in \text{assign}(\mathcal{M}) \text{ such that } \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha) \text{ is defined [designated } C(\ulcorner \mathbf{D}_\delta \urcorner)].$$

Our proof is by induction on the complexity of  $\mathbf{D}_\delta$ . There are 9 cases corresponding to the 9 parts of Specification 9 used to specify  $S(\ulcorner \mathbf{D}_\delta \urcorner)$ . Let  $\varphi \in \text{assign}(\mathcal{M})$  such that  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)$  is defined.

**Case 1:**  $\mathbf{D}_\delta$  is  $\mathbf{x}_\alpha$ . Then

$$\mathcal{V}_\varphi^{\mathcal{M}}(\llbracket S(\ulcorner \mathbf{x}_\alpha \urcorner) \rrbracket_\alpha) \quad (1)$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \text{cleanse}_{\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \rrbracket_\alpha) \quad (2)$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha) \quad (3)$$

$$\simeq \mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)]}^{\mathcal{M}}(\mathbf{x}_\alpha). \quad (4)$$

(2) is by the specification of  $\text{sub}_{\epsilon\epsilon\epsilon}$ ; (3) is by  $H(\ulcorner \mathbf{x}_\alpha \urcorner)$  and part 4 of Lemma 7.4.1; and (4) is by the semantics of variables. Therefore,  $C(\ulcorner \mathbf{x}_\alpha \urcorner)$  holds.

**Case 2:**  $\mathbf{D}_\delta$  is  $\mathbf{y}_\beta$  where  $\mathbf{x}_\alpha \neq \mathbf{y}_\beta$ . Then

$$\mathcal{V}_\varphi^{\mathcal{M}}(\llbracket S(\ulcorner \mathbf{y}_\beta \urcorner) \rrbracket_\beta) \quad (1)$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \ulcorner \mathbf{y}_\beta \urcorner \rrbracket_\beta) \quad (2)$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{y}_\beta) \quad (3)$$

$$\simeq \mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)]}^{\mathcal{M}}(\mathbf{y}_\beta). \quad (4)$$

(2) is by the specification of  $\text{sub}_{\epsilon\epsilon\epsilon}$ ; (3) is by the semantics of evaluation and that fact that  $\mathbf{y}_\beta$  is evaluation-free; and (4) follows from  $\mathbf{x}_\alpha \neq \mathbf{y}_\beta$ . Therefore,  $C(\ulcorner \mathbf{y}_\beta \urcorner)$  holds.

**Case 3:**  $\mathbf{D}_\delta$  is a primitive constant  $\mathbf{c}_\beta$ . Similar to Case 2.

**Case 4:**  $\mathbf{D}_\delta$  is  $\mathbf{B}_{\beta\gamma}\mathbf{D}_\delta$ .  $H(\ulcorner \mathbf{B}_{\beta\gamma}\mathbf{D}_\delta \urcorner)$  implies  $H(\ulcorner \mathbf{B}_{\beta\gamma} \urcorner)$  and  $H(\ulcorner \mathbf{D}_\delta \urcorner)$  by the specification of  $\text{sub}_{\epsilon\epsilon\epsilon}$ . These imply  $C(\ulcorner \mathbf{B}_{\beta\gamma} \urcorner)$  and  $C(\ulcorner \mathbf{D}_\delta \urcorner)$  by the induction hypothesis. Then

$$\mathcal{V}_\varphi^{\mathcal{M}}(\llbracket S(\ulcorner \mathbf{B}_{\beta\gamma}\mathbf{D}_\delta \urcorner) \rrbracket_\beta) \quad (1)$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \text{app}_{\epsilon\epsilon\epsilon} S(\ulcorner \mathbf{B}_{\beta\gamma} \urcorner) S(\ulcorner \mathbf{D}_\delta \urcorner) \rrbracket_\alpha) \quad (2)$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket S(\ulcorner \mathbf{B}_{\beta\gamma} \urcorner) \rrbracket_{\beta\gamma} \llbracket S(\ulcorner \mathbf{D}_\delta \urcorner) \rrbracket_\gamma) \quad (3)$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket S(\ulcorner \mathbf{B}_{\beta\gamma} \urcorner) \rrbracket_{\beta\gamma}) (\mathcal{V}_\varphi^{\mathcal{M}}(\llbracket S(\ulcorner \mathbf{D}_\delta \urcorner) \rrbracket_\gamma)) \quad (4)$$

$$\simeq \mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)]}^{\mathcal{M}}(\mathbf{B}_{\beta\gamma}) (\mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)]}^{\mathcal{M}}(\mathbf{D}_\delta)) \quad (5)$$

$$\simeq \mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)]}^{\mathcal{M}}(\mathbf{B}_{\beta\gamma}\mathbf{D}_\delta). \quad (6)$$

(2) is by the specification of  $\text{sub}_{\epsilon\epsilon\epsilon}$ ; (3) is by the semantics of  $\text{app}_{\epsilon\epsilon\epsilon}$  and evaluation; (4) and (6) are by the semantics of application; and (5) is by  $C(\ulcorner \mathbf{B}_{\beta\gamma} \urcorner)$  and  $C(\ulcorner \mathbf{D}_\delta \urcorner)$ . Therefore,  $C(\ulcorner \mathbf{B}_{\beta\gamma}\mathbf{D}_\delta \urcorner)$  holds.

**Case 5:**  $\mathbf{D}_\delta$  is  $\lambda\mathbf{x}_\alpha\mathbf{B}_\beta$ .  $H(\ulcorner \lambda\mathbf{x}_\alpha\mathbf{B}_\beta \urcorner)$  implies  $\mathcal{M} \models [\text{cleanse}_{\epsilon\epsilon} \ulcorner \mathbf{B}_\beta \urcorner] \downarrow$  by the specification of  $\text{sub}_{\epsilon\epsilon\epsilon}$ . This implies that  $\text{cleanse}_{\epsilon\epsilon} \ulcorner \mathbf{B}_\beta \urcorner$  is semantically closed by part 1 of Lemma 7.4.1. Then

$$\mathcal{V}_\varphi^{\mathcal{M}}(\llbracket S(\lambda\mathbf{x}_\alpha\mathbf{B}_\beta) \rrbracket_{\beta\alpha}) \quad (1)$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \text{abs}_{\epsilon\epsilon\epsilon} \ulcorner \mathbf{x}_\alpha \urcorner \text{cleanse}_{\epsilon\epsilon} \ulcorner \mathbf{B}_\beta \urcorner \rrbracket_{\beta\alpha}) \quad (2)$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\lambda\mathbf{x}_\alpha \llbracket \text{cleanse}_{\epsilon\epsilon} \ulcorner \mathbf{B}_\beta \urcorner \rrbracket_\beta) \quad (3)$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\lambda\mathbf{x}_\alpha\mathbf{B}_\beta) \quad (4)$$

$$\simeq \mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)]}^{\mathcal{M}}(\lambda\mathbf{x}_\alpha\mathbf{B}_\beta). \quad (5)$$

(2) is by the specification of  $\text{sub}_{\epsilon\epsilon\epsilon}$ ; (3) is by the semantics of  $\text{abs}_{\epsilon\epsilon\epsilon}$  and evaluation and the fact that  $\text{cleanse}_{\epsilon\epsilon} \ulcorner \mathbf{B}_\beta \urcorner$  is semantically closed; (4) is by part 4 of Lemma 7.4.1; and (5) is by the fact that

$$\mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto d]}^{\mathcal{M}}(\mathbf{B}_\beta) \simeq \mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)][\mathbf{x}_\alpha \mapsto d]}^{\mathcal{M}}(\mathbf{B}_\beta)$$

for all  $d \in \mathcal{D}_\alpha$ . Therefore,  $C(\ulcorner \lambda\mathbf{x}_\alpha\mathbf{B}_\beta \urcorner)$  holds.

**Case 6:**  $\mathbf{D}_\delta$  is  $\lambda \mathbf{y}_\beta \mathbf{B}_\gamma$  where  $\mathbf{x}_\alpha \neq \mathbf{y}_\beta$ .  $H(\ulcorner \lambda \mathbf{y}_\beta \mathbf{B}_\gamma \urcorner)$  implies

$$(a) \mathcal{V}_\varphi^{\mathcal{M}}(S(\ulcorner \lambda \mathbf{y}_\beta \mathbf{B}_\gamma \urcorner)) \simeq \mathcal{V}_\varphi^{\mathcal{M}}(\text{abs}_{\epsilon\epsilon\epsilon} \ulcorner \mathbf{y}_\beta \urcorner S(\ulcorner \mathbf{B}_\gamma \urcorner)),$$

$H(\ulcorner \mathbf{B}_\gamma \urcorner)$ , and either  $(*) \mathcal{M} \models \text{not-free-in}_{o\epsilon\epsilon} \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\gamma \urcorner$  or  $(**) \mathcal{M} \models \text{not-free-in}_{o\epsilon\epsilon} \ulcorner \mathbf{y}_\beta \urcorner \ulcorner \mathbf{A}_\alpha \urcorner$  by the specification of  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$ .  $H(\ulcorner \mathbf{B}_\gamma \urcorner)$  implies  $C(\ulcorner \mathbf{B}_\gamma \urcorner)$  by the induction hypothesis and (b)  $S(\ulcorner \mathbf{B}_\gamma \urcorner)$  is semantically closed by part 1 of this lemma. Then

$$\mathcal{V}_\varphi^{\mathcal{M}}(\llbracket S(\lambda \mathbf{y}_\beta \mathbf{B}_\gamma) \rrbracket_{\gamma\beta}) \quad (1)$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \text{abs}_{\epsilon\epsilon\epsilon} \ulcorner \mathbf{y}_\beta \urcorner S(\ulcorner \mathbf{B}_\gamma \urcorner) \rrbracket_{\gamma\beta}) \quad (2)$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\lambda \mathbf{y}_\beta \llbracket S(\ulcorner \mathbf{B}_\gamma \urcorner) \rrbracket_\gamma) \quad (3)$$

$$\simeq \mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)]}^{\mathcal{M}}(\lambda \mathbf{y}_\beta \mathbf{B}_\gamma). \quad (4)$$

(2) is by (a); (3) is by (b) and the semantics of  $\text{abs}_{\epsilon\epsilon\epsilon}$  and evaluation; and (4) is by separate arguments for the two cases  $(*)$  and  $(**)$ . In case  $(*)$ ,

$$\mathcal{V}_{\varphi[\mathbf{y}_\alpha \mapsto d]}^{\mathcal{M}}(\llbracket S(\ulcorner \mathbf{B}_\gamma \urcorner) \rrbracket_\gamma) \quad (1)$$

$$\simeq \mathcal{V}_{\varphi[\mathbf{y}_\alpha \mapsto d]}^{\mathcal{M}}(\mathbf{B}_\gamma) \quad (2)$$

$$\simeq \mathcal{V}_{\varphi[\mathbf{y}_\alpha \mapsto d][\mathbf{x}_\alpha \mapsto \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)]}^{\mathcal{M}}(\mathbf{B}_\gamma) \quad (3)$$

$$\simeq \mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)][\mathbf{y}_\alpha \mapsto d]}^{\mathcal{M}}(\mathbf{B}_\gamma) \quad (4)$$

for all  $d \in \mathcal{D}_\alpha$ . (2) is by  $(*)$ ,  $H(\ulcorner \mathbf{B}_\gamma \urcorner)$ , and part 5 of this lemma; (3) is by  $(*)$  and part 2 of Lemma 7.3.1; and (4) follows from  $\mathbf{x}_\alpha \neq \mathbf{y}_\beta$ . In case  $(**)$ ,

$$\mathcal{V}_{\varphi[\mathbf{y}_\alpha \mapsto d]}^{\mathcal{M}}(\llbracket S(\ulcorner \mathbf{B}_\gamma \urcorner) \rrbracket_\gamma) \quad (1)$$

$$\simeq \mathcal{V}_{\varphi[\mathbf{y}_\alpha \mapsto d][\mathbf{x}_\alpha \mapsto \mathcal{V}_{\varphi[\mathbf{y}_\alpha \mapsto d]}^{\mathcal{M}}(\mathbf{A}_\alpha)]}^{\mathcal{M}}(\mathbf{B}_\gamma) \quad (2)$$

$$\simeq \mathcal{V}_{\varphi[\mathbf{y}_\alpha \mapsto d][\mathbf{x}_\alpha \mapsto \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)]}^{\mathcal{M}}(\mathbf{B}_\gamma) \quad (3)$$

$$\simeq \mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)][\mathbf{y}_\alpha \mapsto d]}^{\mathcal{M}}(\mathbf{B}_\gamma) \quad (4)$$

for all  $d \in \mathcal{D}_\alpha$ . (2) is by  $C(\ulcorner \mathbf{B}_\gamma \urcorner)$ ; (3) is by  $(**)$  and part 2 of Lemma 7.3.1; and (4) follows from  $\mathbf{x}_\alpha \neq \mathbf{y}_\beta$ . Therefore,  $C(\ulcorner \lambda \mathbf{y}_\beta \mathbf{B}_\gamma \urcorner)$  holds.

**Case 7:**  $\mathbf{D}_\delta$  is if  $\mathbf{A}_o \mathbf{B}_\alpha \mathbf{C}_\alpha$ . Similar to Case 4.

**Case 8:**  $\mathbf{D}_\delta$  is  $\ulcorner \mathbf{B}_\beta \urcorner$ . Similar to Case 2.

**Case 9:**  $\mathbf{D}_\delta$  is  $\llbracket \mathbf{B}_\epsilon \rrbracket_\beta$ .  $H(\ulcorner \llbracket \mathbf{B}_\epsilon \rrbracket_\beta \urcorner)$  implies

$$(a) \mathcal{M} \models \text{eval-free}_{o\epsilon}^\beta \llbracket S(\ulcorner \mathbf{B}_\epsilon \urcorner) \rrbracket_\epsilon$$

and

$$(b) \mathcal{M} \models S(\ulcorner \llbracket \mathbf{B}_\epsilon \rrbracket_\beta \urcorner) = S(\llbracket S(\ulcorner \mathbf{B}_\epsilon \urcorner) \rrbracket_\epsilon)$$

by the specification of  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$ . (a), (b), and  $H(\ulcorner \llbracket \mathbf{B}_\epsilon \rrbracket_\beta \urcorner)$  imply  $H(\llbracket S(\ulcorner \mathbf{B}_\epsilon \urcorner) \rrbracket_\epsilon)$  by part 3 of this lemma, and so (c)  $\mathcal{M} \models \llbracket S(\ulcorner \mathbf{B}_\epsilon \urcorner) \rrbracket_\epsilon = \ulcorner \mathbf{C}_\beta \urcorner$  for some evaluation-free  $\mathbf{C}_\beta$ .  $H(\llbracket S(\ulcorner \mathbf{B}_\epsilon \urcorner) \rrbracket_\epsilon)$  implies  $H(\ulcorner \mathbf{B}_\epsilon \urcorner)$  by the specification of  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$ .  $H(\ulcorner \mathbf{B}_\epsilon \urcorner)$  implies  $C(\ulcorner \mathbf{B}_\epsilon \urcorner)$  by the induction hypothesis, and so

$$(d) \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket S(\ulcorner \mathbf{B}_\epsilon \urcorner) \rrbracket_\epsilon) \simeq \mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)]}^{\mathcal{M}}(\mathbf{B}_\epsilon).$$

(c) and (d) imply

$$(e) \mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)]}^{\mathcal{M}}(\mathbf{B}_\epsilon) = \mathcal{V}_\varphi^{\mathcal{M}}(\ulcorner \mathbf{C}_\beta \urcorner) = \mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)]}^{\mathcal{M}}(\ulcorner \mathbf{C}_\beta \urcorner)$$

since  $\ulcorner \mathbf{C}_\beta \urcorner$  is semantically closed.  $H(\llbracket S(\ulcorner \mathbf{B}_\epsilon \urcorner) \rrbracket_\epsilon)$  and (c) imply  $H(\ulcorner \mathbf{C}_\beta \urcorner)$ , and  $H(\ulcorner \mathbf{C}_\beta \urcorner)$  implies  $C(\ulcorner \mathbf{C}_\beta \urcorner)$  by the inductive hypothesis.

$$\mathcal{V}_\varphi^{\mathcal{M}}(\llbracket S(\ulcorner \llbracket \mathbf{B}_\epsilon \rrbracket_\beta \urcorner) \rrbracket_\beta) \tag{1}$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket S(\llbracket S(\ulcorner \mathbf{B}_\epsilon \urcorner) \rrbracket_\epsilon) \rrbracket_\beta) \tag{2}$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket S(\ulcorner \mathbf{C}_\beta \urcorner) \rrbracket_\beta) \tag{3}$$

$$\simeq \mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)]}^{\mathcal{M}}(\mathbf{C}_\beta) \tag{4}$$

$$\simeq \mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)]}^{\mathcal{M}}(\llbracket \ulcorner \mathbf{C}_\beta \urcorner \rrbracket_\beta) \tag{5}$$

$$\simeq \mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)]}^{\mathcal{M}}(\llbracket \mathbf{B}_\epsilon \rrbracket_\beta). \tag{6}$$

(2) is by (b); (3) is by (d) and (e); (4) is by  $C(\ulcorner \mathbf{C}_\beta \urcorner)$ ; (5) is by the semantics of evaluation and the fact  $\mathbf{C}_\beta$  is evaluation-free; and (6) is by (e). Therefore,  $C(\ulcorner \llbracket \mathbf{B}_\epsilon \rrbracket_\beta \urcorner)$  holds.

**Part 7** Follows from the specifications of  $\text{not-free-in}_{o\epsilon\epsilon}$ ,  $\text{cleanse}_{\epsilon\epsilon}$ , and  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$ .  $\square$

The five requirements for  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$  are satisfied as follows:

1. Requirement 1 is satisfied by Specification 9 for  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$ . Part 6 of Lemma 7.5.1 verifies that  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$  performs substitution correctly.
2. Requirement 2 is satisfied by Specification 7 for  $\text{not-free-in}_{o\epsilon\epsilon}$  and Specification 9.6 for  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$ . Part 6 of Lemma 7.5.1 verifies that, when  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner$  is defined, variables are not captured.
3. Requirement 3 is satisfied by Specification 8 for  $\text{cleanse}_{\epsilon\epsilon}$  and Specifications 9.1, 9.5, and 9.9 for  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$ . Part 1 of Lemma 7.5.1 verifies that, when  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner$  is defined, it represents an evaluation-free wff  $\beta$ .
4. Requirement 4 is satisfied by Specification 7.9 for  $\text{not-free-in}_{o\epsilon\epsilon}$ , Specification 8.7 for  $\text{cleanse}_{\epsilon\epsilon}$ , and Specification 9.9 for  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$ . Part 1 of Lemma 7.5.1 verifies that, when  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner$  is defined, it is semantically closed.



5. Requirement 5 is satisfied by Specifications 7–9. Part 7 of Lemma 7.5.1 verifies that  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner$  is defined in the cases corresponding to when substitution is defined in  $\mathcal{Q}_0$ .

As a consequence of  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$  satisfying Requirements 1–5, we can now prove that the law of beta-reduction for  $\mathcal{Q}_0^{\text{uqe}}$  is valid in  $\mathcal{Q}_0^{\text{uqe}}$ :

**Theorem 7.5.2 (Law of Beta-Reduction)** *Let  $\mathcal{M}$  be a normal general model for  $\mathcal{Q}_0^{\text{uqe}}$ . Then*

$$\mathcal{M} \models [\mathbf{A}_\alpha \downarrow \wedge \text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner = \ulcorner \mathbf{C}_\beta \urcorner] \supset [\lambda \mathbf{x}_\alpha \mathbf{B}_\beta] \mathbf{A}_\alpha \simeq \mathbf{C}_\beta.$$

**Proof** Let  $\varphi \in \text{assign}(\mathcal{M})$ . Assume (a)  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)$  is defined and

$$(b) \mathcal{V}_\varphi^{\mathcal{M}}(\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner = \ulcorner \mathbf{C}_\beta \urcorner) = \top.$$

We must show

$$\mathcal{V}_\varphi^{\mathcal{M}}([\lambda \mathbf{x}_\alpha \mathbf{B}_\beta] \mathbf{A}_\alpha) \simeq \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{C}_\beta).$$

(b) implies

$$(c) \mathcal{M} \models \text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner = \ulcorner \mathbf{C}_\beta \urcorner$$

and (d)  $\mathbf{C}_\beta$  is evaluation-free by part 1 of Lemma 7.5.1. Then

$$\mathcal{V}_\varphi^{\mathcal{M}}([\lambda \mathbf{x}_\alpha \mathbf{B}_\beta] \mathbf{A}_\alpha) \tag{1}$$

$$\simeq \mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)]}^{\mathcal{M}}(\mathbf{B}_\beta) \tag{2}$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner \rrbracket_\beta) \tag{3}$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \ulcorner \mathbf{C}_\beta \urcorner \rrbracket_\beta) \tag{4}$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{C}_\beta). \tag{5}$$

(2) is by (a) and the semantics of function application and function abstraction; (3) is by (c) and part 6 of Lemma 7.5.1; (4) is by (b); and (5) is by (d) and the semantics of evaluation. Therefore,  $\mathcal{V}_\varphi^{\mathcal{M}}([\lambda \mathbf{x}_\alpha \mathbf{B}_\beta] \mathbf{A}_\alpha) \simeq \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{C}_\beta)$ .  $\square$

## 7.6 Example: Double Substitution

We mentioned above that  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner$  may involve a “double substitution” when  $\mathbf{B}_\beta$  is an evaluation. The following example explores this possibility when  $\mathbf{B}_\beta$  is the simple evaluation  $\llbracket \mathbf{x}_\epsilon \rrbracket_o$ .

Let  $\mathcal{M}$  be any normal general model for  $\mathcal{Q}_0^{\text{uqe}}$ ,  $\varphi \in \text{assign}(\mathcal{M})$ , and  $\mathbf{A}_o$  be an evaluation-free wff in which  $\mathbf{x}_\epsilon$  is not free. Then

$$\mathcal{V}_\varphi^{\mathcal{M}}(\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_o \urcorner \ulcorner \mathbf{x}_\epsilon \urcorner \ulcorner \llbracket \mathbf{x}_\epsilon \rrbracket_o \urcorner) \tag{1}$$

$$= \mathcal{V}_\varphi^{\mathcal{M}}(\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_o \urcorner \ulcorner \mathbf{x}_\epsilon \urcorner \ulcorner \llbracket \text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_o \urcorner \ulcorner \mathbf{x}_\epsilon \urcorner \ulcorner \mathbf{x}_\epsilon \urcorner \rrbracket_\epsilon \urcorner) \tag{2}$$

$$= \mathcal{V}_\varphi^{\mathcal{M}}(\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_o \urcorner \ulcorner \mathbf{x}_\epsilon \urcorner \ulcorner \llbracket \ulcorner \mathbf{A}_o \urcorner \rrbracket_\epsilon \urcorner) \tag{3}$$

$$= \mathcal{V}_\varphi^{\mathcal{M}}(\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_o \urcorner \ulcorner \mathbf{x}_\epsilon \urcorner \ulcorner \ulcorner \mathbf{A}_o \urcorner \urcorner) \tag{4}$$

$$= \mathcal{V}_\varphi^{\mathcal{M}}(\ulcorner \mathbf{A}_o \urcorner). \tag{5}$$

(2) is by the specification of  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$ , the fact that  $\ulcorner \mathbf{A}_o \urcorner$  is syntactically closed, and the fact that  $\mathbf{A}_o$  is evaluation-free; (3) is by the specification of  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$ ; (4) is by the semantics of evaluation and the fact that  $\ulcorner \mathbf{A}_o \urcorner$  is evaluation-free; and (5) is by the specification of  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$  and the fact that  $\mathbf{x}_\epsilon$  is not free in  $\mathbf{A}_o$ . Therefore,

$$\mathcal{M} \models \text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \ulcorner \mathbf{A}_o \urcorner \urcorner \ulcorner \mathbf{x}_\epsilon \urcorner \ulcorner \llbracket \mathbf{x}_\epsilon \rrbracket_o \urcorner \urcorner = \ulcorner \mathbf{A}_o \urcorner$$

and only the first substitution has an effect.

Now consider the evaluation-free wff  $\mathbf{x}_\epsilon = \mathbf{x}_\epsilon$  (in which the variable  $\mathbf{x}_\epsilon$  is free). Then

$$\mathcal{V}_\varphi^{\mathcal{M}}(\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \ulcorner \mathbf{x}_\epsilon = \mathbf{x}_\epsilon \urcorner \urcorner \ulcorner \mathbf{x}_\epsilon \urcorner \ulcorner \llbracket \mathbf{x}_\epsilon \rrbracket_o \urcorner \urcorner) \quad (1)$$

$$= \mathcal{V}_\varphi^{\mathcal{M}}(\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \ulcorner \mathbf{x}_\epsilon = \mathbf{x}_\epsilon \urcorner \urcorner \ulcorner \mathbf{x}_\epsilon \urcorner \ulcorner \llbracket \text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \ulcorner \mathbf{x}_\epsilon = \mathbf{x}_\epsilon \urcorner \urcorner \ulcorner \mathbf{x}_\epsilon \urcorner \ulcorner \llbracket \mathbf{x}_\epsilon \rrbracket_o \urcorner \urcorner \urcorner \urcorner) \quad (2)$$

$$= \mathcal{V}_\varphi^{\mathcal{M}}(\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \ulcorner \mathbf{x}_\epsilon = \mathbf{x}_\epsilon \urcorner \urcorner \ulcorner \mathbf{x}_\epsilon \urcorner \ulcorner \llbracket \ulcorner \mathbf{x}_\epsilon = \mathbf{x}_\epsilon \urcorner \urcorner \urcorner \urcorner) \quad (3)$$

$$= \mathcal{V}_\varphi^{\mathcal{M}}(\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \ulcorner \mathbf{x}_\epsilon = \mathbf{x}_\epsilon \urcorner \urcorner \ulcorner \mathbf{x}_\epsilon \urcorner \ulcorner \llbracket \mathbf{x}_\epsilon = \mathbf{x}_\epsilon \urcorner \urcorner \urcorner) \quad (4)$$

$$= \mathcal{V}_\varphi^{\mathcal{M}}(\ulcorner \ulcorner \mathbf{x}_\epsilon = \mathbf{x}_\epsilon \urcorner \urcorner = \ulcorner \mathbf{x}_\epsilon = \mathbf{x}_\epsilon \urcorner \urcorner). \quad (5)$$

(1)–(4) are by the same reasoning as above, and (5) is by the specification of  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$ . Therefore,

$$\mathcal{M} \models \text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \ulcorner \mathbf{x}_\epsilon = \mathbf{x}_\epsilon \urcorner \urcorner \ulcorner \mathbf{x}_\epsilon \urcorner \ulcorner \llbracket \mathbf{x}_\epsilon \rrbracket_o \urcorner \urcorner = \ulcorner \ulcorner \mathbf{x}_\epsilon = \mathbf{x}_\epsilon \urcorner \urcorner = \ulcorner \mathbf{x}_\epsilon = \mathbf{x}_\epsilon \urcorner \urcorner$$

and both substitutions have an effect.

## 7.7 Example: Variable Renaming

In predicate logics like  $\mathcal{Q}_0$ , bound variables can be renamed in a wff (in certain ways) without changing the meaning the wff. For example, when the variable  $x_\alpha$  is renamed to the variable  $y_\alpha$  (or any other variable of type  $\alpha$ ) in the evaluation-free wff  $\lambda x_\alpha x_\alpha$ , the result is the wff  $\lambda y_\alpha y_\alpha$ .  $\lambda x_\alpha x_\alpha$  and  $\lambda y_\alpha y_\alpha$  are logically equivalent to each other, i.e.,

$$\mathcal{M} \models \lambda x_\alpha x_\alpha = \lambda y_\alpha y_\alpha.$$

In fact, a variable renaming that permutes the names of the variables occurring in an evaluation-free wff of  $\mathcal{Q}_0^{\text{uqe}}$  without changing the names of the wff's free variables preserves the meaning of the wff.

Unfortunately, meaning-preserving variable renamings do not exist for all the non-evaluation-free wffs of  $\mathcal{Q}_0^{\text{uqe}}$ . As an example, consider the two non-evaluation-free wffs  $\lambda x_\epsilon \llbracket x_\epsilon \rrbracket_{\langle \epsilon \epsilon \rangle}$  and  $\lambda y_\epsilon \llbracket y_\epsilon \rrbracket_{\langle \epsilon \epsilon \rangle}$  where  $x_\epsilon$  and  $y_\epsilon$  are distinct variables. Obviously,  $\lambda y_\epsilon \llbracket y_\epsilon \rrbracket_{\langle \epsilon \epsilon \rangle}$  is obtained from  $\lambda x_\epsilon \llbracket x_\epsilon \rrbracket_{\langle \epsilon \epsilon \rangle}$  by renaming  $x_\epsilon$  to be  $y_\epsilon$ . If we forget about evaluation, we would expect that  $\lambda x_\epsilon \llbracket x_\epsilon \rrbracket_{\langle \epsilon \epsilon \rangle}$  and  $\lambda y_\epsilon \llbracket y_\epsilon \rrbracket_{\langle \epsilon \epsilon \rangle}$  are logically equivalent — but they are not! Let  $\mathbf{A}_\epsilon$  be

$\ulcorner \text{pair}_{\langle \epsilon \rangle \epsilon \epsilon} x_\epsilon y_\epsilon \urcorner$ , and suppose  $\varphi(x_\epsilon) = \mathcal{E}(x_\epsilon)$  and  $\varphi(y_\epsilon) = \mathcal{E}(y_\epsilon)$ . Then

$$\begin{aligned}
& \mathcal{V}_\varphi^{\mathcal{M}}([\lambda x_\epsilon \llbracket x_\epsilon \rrbracket_{\langle \epsilon \rangle} ] \mathbf{A}_\epsilon) \\
& \simeq \mathcal{V}_{\varphi[x_\epsilon \mapsto \mathcal{V}_{\varphi[\mathbf{A}_\epsilon]}]}^{\mathcal{M}}(\llbracket x_\epsilon \rrbracket_{\langle \epsilon \rangle}) \\
& \simeq \mathcal{V}_{\varphi[x_\epsilon \mapsto \mathcal{V}_{\varphi[\mathbf{A}_\epsilon]}]}^{\mathcal{M}}(\mathcal{E}^{-1}(\mathcal{V}_{\varphi[x_\epsilon \mapsto \mathcal{V}_{\varphi[\mathbf{A}_\epsilon]}]}^{\mathcal{M}}(x_\epsilon))) \\
& \simeq \mathcal{V}_{\varphi[x_\epsilon \mapsto \mathcal{V}_{\varphi[\mathbf{A}_\epsilon]}]}^{\mathcal{M}}(\mathcal{E}^{-1}(\mathcal{E}(\text{pair}_{\langle \epsilon \rangle \epsilon \epsilon} x_\epsilon y_\epsilon))) \\
& \simeq \mathcal{V}_{\varphi[x_\epsilon \mapsto \mathcal{V}_{\varphi[\mathbf{A}_\epsilon]}]}^{\mathcal{M}}(\text{pair}_{\langle \epsilon \rangle \epsilon \epsilon} x_\epsilon y_\epsilon) \\
& = \langle \mathcal{E}(\text{pair}_{\langle \epsilon \rangle \epsilon \epsilon} x_\epsilon y_\epsilon), \mathcal{E}(y_\epsilon) \rangle.
\end{aligned}$$

Similarly,

$$\mathcal{V}_\varphi^{\mathcal{M}}([\lambda y_\epsilon \llbracket y_\epsilon \rrbracket_{\langle \epsilon \rangle} ] \mathbf{A}_\epsilon) \simeq \langle \mathcal{E}(x_\epsilon), \mathcal{E}(\text{pair}_{\langle \epsilon \rangle \epsilon \epsilon} x_\epsilon y_\epsilon) \rangle.$$

Therefore,  $\lambda x_\epsilon \llbracket x_\epsilon \rrbracket_{\langle \epsilon \rangle}$  and  $\lambda y_\epsilon \llbracket y_\epsilon \rrbracket_{\langle \epsilon \rangle}$  are not logically equivalent, but the functions  $\mathcal{V}_\varphi^{\mathcal{M}}(\lambda x_\epsilon \llbracket x_\epsilon \rrbracket_{\langle \epsilon \rangle})$  and  $\mathcal{V}_\varphi^{\mathcal{M}}(\lambda y_\epsilon \llbracket y_\epsilon \rrbracket_{\langle \epsilon \rangle})$  are equal on constructions of the form  $\mathcal{E}(B_{\langle \epsilon \rangle})$  where  $B_{\langle \epsilon \rangle}$  is semantically closed.

This example proves the following proposition:

**Proposition 7.7.1** *Alpha-conversion is not valid in  $\mathcal{Q}_0^{\text{uqe}}$  for some non-evaluation-free wffs.*

**Note 19 (Nominal Data Types)** Since alpha-conversion is not universally valid in  $\mathcal{Q}_0^{\text{uqe}}$ , it is not clear whether techniques for managing variable naming and binding — such as *higher-order abstract syntax* [46, 52] and *nominal techniques* [29, 53] — are applicable to  $\mathcal{Q}_0^{\text{uqe}}$ . However, the paper [48] does combine quotation/evaluation techniques with nominal techniques.

## 7.8 Limitations of $\text{sub}_{\epsilon \epsilon \epsilon \epsilon}$

Theorem 7.5.2 shows beta-reduction can be computed using  $\text{sub}_{\epsilon \epsilon \epsilon \epsilon}$ . However, it is obviously not possible to use  $\text{sub}_{\epsilon \epsilon \epsilon \epsilon}$  to compute a beta-reduction when the corresponding application of  $\text{sub}_{\epsilon \epsilon \epsilon \epsilon}$  is undefined. There are thus two questions that concern us:

1. When is an application of  $\text{sub}_{\epsilon \epsilon \epsilon \epsilon}$  undefined?
2. When an application of  $\text{sub}_{\epsilon \epsilon \epsilon \epsilon}$  is undefined, is the the corresponding beta-reduction ever valid in  $\mathcal{Q}_0^{\text{uqe}}$ .

Let  $\mathcal{M}$  be a normal general model for  $\mathcal{Q}_0^{\text{uqe}}$ . There are two cases in which  $\mathcal{M} \models [\text{sub}_{\epsilon \epsilon \epsilon \epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner ] \uparrow$  will be true. The first case occurs when the naive substitution of  $\mathbf{A}_\alpha$  for the free occurrences of  $\mathbf{x}_\alpha$  in  $\mathbf{B}_\beta$  causes a variable capture. In this case the corresponding beta-reduction is not valid unless the bound variables in  $\mathbf{B}_\beta$  are renamed so that the variable capture is avoided. This can always be done if  $\mathbf{B}_\beta$  is evaluation-free, but as we showed in the previous

subsection it is not always possible to rename variables in a non-evaluation-free wff.

The second case in which  $\mathcal{M} \models [\text{sub}_{\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner ] \uparrow$  will be true occurs when the naive cleansing of evaluations in the result of the substitution causes a variable to escape outside of a quotation. This happens when the body of an evaluation is not semantically closed after the first substitution. In this case, the corresponding beta-reduction may be valid. We will illustrate this possibility with three examples.

### Example 1

Let  $\mathbf{A}_{\alpha\epsilon\epsilon}$  be the wff

$$\lambda x_\epsilon \lambda y_\epsilon \llbracket \text{app}_{\epsilon\epsilon\epsilon} x_\epsilon y_\epsilon \rrbracket_\alpha$$

and  $\mathbf{B}_{\alpha\beta}$  and  $\mathbf{C}_\beta$  be syntactically closed evaluation-free wffs. Then

$$\mathcal{M} \models \mathbf{A}_{\alpha\epsilon\epsilon} \ulcorner \mathbf{B}_{\alpha\beta} \urcorner \ulcorner \mathbf{C}_\beta \urcorner \simeq \mathbf{B}_{\alpha\beta} \mathbf{C}_\beta.$$

However, we also have

$$\mathcal{M} \models [\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{B}_{\alpha\beta} \urcorner \ulcorner x_\epsilon \urcorner \ulcorner \mathbf{A}_{\alpha\epsilon\epsilon} \urcorner ] \uparrow$$

since the body of the evaluation contains  $y_\epsilon$  after the first substitution. Hence the beta-reduction of  $\mathbf{A}_{\alpha\epsilon\epsilon} \ulcorner \mathbf{B}_{\alpha\beta} \urcorner \ulcorner \mathbf{C}_\beta \urcorner$  is valid in  $\mathcal{Q}_0^{\text{uqe}}$ , but the corresponding application of  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$  is undefined.

This is a significant limitation. It means, for instance, that using  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$  we cannot instantiate a formula with more than one variable within an evaluation (not in a quotation). An instance of specification 9.9 where the syntactic variables are replaced with variables is an example of a formula with this property.

In some cases this limitation can be overcome by instantiating all the variables of type  $\epsilon$  within an evaluation together as a group. For example, let  $\mathbf{A}'_{\alpha\epsilon\epsilon}$  be the wff  $\lambda x_{\langle\epsilon\epsilon\rangle} \mathbf{D}_\alpha$  where  $\mathbf{D}_\alpha$  is

$$\llbracket \text{app}_{\epsilon\epsilon\epsilon} [\text{fst}_{\epsilon\langle\epsilon\epsilon\rangle} x_{\langle\epsilon\epsilon\rangle}] [\text{snd}_{\epsilon\langle\epsilon\epsilon\rangle} x_{\langle\epsilon\epsilon\rangle}] \rrbracket_\alpha.$$

Then

$$\begin{aligned} & \mathcal{V}_\varphi^{\mathcal{M}}(\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \text{pair}_{\langle\epsilon\epsilon\rangle\epsilon\epsilon} \ulcorner \mathbf{B}_{\alpha\beta} \urcorner \ulcorner \mathbf{C}_\beta \urcorner \urcorner \ulcorner x_{\langle\epsilon\epsilon\rangle} \urcorner \ulcorner \mathbf{D}_\alpha \urcorner \urcorner) \\ & \simeq \mathcal{V}_\varphi^{\mathcal{M}}(\text{app}_{\epsilon\epsilon\epsilon} [\text{fst}_{\epsilon\langle\epsilon\epsilon\rangle} [\text{pair}_{\langle\epsilon\epsilon\rangle\epsilon\epsilon} \ulcorner \mathbf{B}_{\alpha\beta} \urcorner \ulcorner \mathbf{C}_\beta \urcorner \urcorner]] \\ & \quad [\text{snd}_{\epsilon\langle\epsilon\epsilon\rangle} [\text{pair}_{\langle\epsilon\epsilon\rangle\epsilon\epsilon} \ulcorner \mathbf{B}_{\alpha\beta} \urcorner \ulcorner \mathbf{C}_\beta \urcorner \urcorner]]) \\ & \simeq \mathcal{V}_\varphi^{\mathcal{M}}(\text{app}_{\epsilon\epsilon\epsilon} \ulcorner \mathbf{B}_{\alpha\beta} \urcorner \ulcorner \mathbf{C}_\beta \urcorner \urcorner) \\ & \simeq \mathcal{V}_\varphi^{\mathcal{M}}(\ulcorner \mathbf{B}_{\alpha\beta} \mathbf{C}_\beta \urcorner) \end{aligned}$$

for all  $\varphi \in \text{assign}(\mathcal{M})$ . Hence

$$\mathcal{M} \models \text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \text{pair}_{\langle\epsilon\epsilon\rangle\epsilon\epsilon} \ulcorner \mathbf{B}_{\alpha\beta} \urcorner \ulcorner \mathbf{C}_\beta \urcorner \urcorner \ulcorner x_\epsilon \urcorner \ulcorner \mathbf{D}_\alpha \urcorner \urcorner = \ulcorner \mathbf{B}_{\alpha\beta} \mathbf{C}_\beta \urcorner,$$

and so by Theorem 7.5.2

$$\mathcal{M} \models \mathbf{A}'_{\alpha\epsilon\epsilon} [\text{pair}_{\langle\epsilon\rangle\epsilon\epsilon} \ulcorner \mathbf{B}_{\alpha\beta} \urcorner \urcorner \mathbf{C}_{\beta} \urcorner] \simeq \mathbf{B}_{\alpha\beta} \mathbf{C}_{\beta}.$$

The main reason we have introduced pairs in  $\mathcal{Q}_0^{\text{uqe}}$  is to allow us to express function abstractions like  $\mathbf{A}_{\alpha\epsilon\epsilon}$  in a form like  $\mathbf{A}'_{\alpha\epsilon\epsilon}$  that can be beta-reduced using  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$ .

### Example 2

Let  $\mathbf{C}_{\alpha}$  be the wff  $[\lambda x_{\epsilon} x_{\epsilon}] [x_{\epsilon}]_{\alpha}$ . Then

$$\mathcal{M} \models [\lambda x_{\epsilon} x_{\epsilon}] [x_{\epsilon}]_{\alpha} \simeq [x_{\epsilon}]_{\alpha}$$

but

$$\mathcal{M} \models [\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner [x_{\epsilon}]_{\alpha} \urcorner \urcorner x_{\epsilon} \urcorner \urcorner x_{\epsilon} \urcorner] \uparrow$$

since  $\mathcal{M} \models [\text{cleanse}_{\epsilon\epsilon} [x_{\epsilon}]_{\alpha}] \uparrow$ . We will overcome this limitation of  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$  by including

$$[\lambda x_{\epsilon} x_{\epsilon}] \mathbf{A}_{\alpha} \simeq \mathbf{A}_{\alpha}$$

and the other basic properties of lambda-notation in the axioms of  $\mathcal{P}^{\text{uqe}}$ . These properties will be presented as schemas similar to Axioms 4<sub>1</sub>–4<sub>5</sub> in [2].

### Example 3

Let  $\mathbf{C}_{\alpha}$  be the wff  $[\lambda x_{\epsilon} [x_{\epsilon}]_{\alpha}] x_{\epsilon}$ . Then

$$\mathcal{M} \models [\lambda x_{\epsilon} [x_{\epsilon}]_{\alpha}] x_{\epsilon} \simeq [x_{\epsilon}]_{\alpha}$$

but

$$\mathcal{M} \models [\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner x_{\epsilon} \urcorner \urcorner x_{\epsilon} \urcorner \urcorner [x_{\epsilon}]_{\alpha} \urcorner] \uparrow$$

since the body of the evaluation contains  $x_{\epsilon}$  after the first substitution. We will overcome this limitation of  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$  by including

$$[\lambda x_{\alpha} \mathbf{B}_{\beta}] x_{\alpha} \simeq \mathbf{B}_{\alpha}$$

in the axioms of  $\mathcal{P}^{\text{uqe}}$ .

## 8 Proof System

Now that we have defined a mechanism for substitution, we are ready to present the proof system of  $\mathcal{Q}_0^{\text{uqe}}$  called  $\mathcal{P}^{\text{uqe}}$ . It is derived from  $\mathcal{P}^{\text{u}}$ , the proof system of  $\mathcal{Q}_0^{\text{u}}$ . The presence of undefinedness makes  $\mathcal{P}^{\text{u}}$  moderately more complicated than  $\mathcal{P}$ , the proof system of  $\mathcal{Q}_0$ , but the presence of the type  $\epsilon$  and quotation and evaluation makes  $\mathcal{P}^{\text{uqe}}$  significantly more complicated than  $\mathcal{P}^{\text{u}}$ . A large part of the complexity of  $\mathcal{Q}_0^{\text{uqe}}$  is due to the difficulty of beta-reducing wffs that involve evaluations.

## 8.1 Axioms

$\mathcal{P}^{\text{ue}}$  consists of a set of axioms and a set of rules of inference. The axioms are given in this section, while the rules of inference are given in the next section. The axioms are organized into groups. The members of each group are presented using one or more formula schemas. A group is called an ‘‘Axiom’’ even though it consists of infinitely many formulas.

### Axiom 1 (Truth Values)

$$[\mathbf{G}_{oo}T_o \wedge \mathbf{G}_{oo}F_o] \equiv \forall \mathbf{x}_o[\mathbf{G}_{oo}\mathbf{x}_o].$$

### Axiom 2 (Leibniz’ Law)

$$\mathbf{A}_\alpha = \mathbf{B}_\alpha \supset [\mathbf{H}_{o\alpha}\mathbf{A}_\alpha \equiv \mathbf{H}_{o\alpha}\mathbf{B}_\alpha].$$

### Axiom 3 (Extensionality)

$$[\mathbf{F}_{\alpha\beta}\downarrow \wedge \mathbf{G}_{\alpha\beta}\downarrow] \supset \mathbf{F}_{\alpha\beta} = \mathbf{G}_{\alpha\beta} \equiv \forall \mathbf{x}_\beta[\mathbf{F}_{\alpha\beta}\mathbf{x}_\beta \simeq \mathbf{G}_{\alpha\beta}\mathbf{x}_\beta].$$

### Axiom 4 (Beta-Reduction)

1.  $[\mathbf{A}_\alpha\downarrow \wedge \text{sub}_{\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner = \ulcorner \mathbf{C}_\beta \urcorner] \supset [\lambda \mathbf{x}_\alpha \mathbf{B}_\beta] \mathbf{A}_\alpha \simeq \mathbf{C}_\beta.$
2.  $[\lambda \mathbf{x}_\alpha \mathbf{x}_\alpha] \mathbf{A}_\alpha \simeq \mathbf{A}_\alpha.$
3.  $\mathbf{A}_\alpha\downarrow \supset [\lambda \mathbf{x}_\alpha \mathbf{y}_\beta] \mathbf{A}_\alpha \simeq \mathbf{y}_\beta$  where  $\mathbf{x}_\alpha \neq \mathbf{y}_\beta.$
4.  $\mathbf{A}_\alpha\downarrow \supset [\lambda \mathbf{x}_\alpha \mathbf{c}_\beta] \mathbf{A}_\alpha \simeq \mathbf{c}_\alpha$  where  $\mathbf{c}_\beta$  is a primitive constant.
5.  $[\lambda \mathbf{x}_\alpha [\mathbf{B}_{\alpha\beta} \mathbf{C}_\beta]] \mathbf{A}_\alpha \simeq [[\lambda \mathbf{x}_\alpha \mathbf{B}_{\alpha\beta}] \mathbf{A}_\alpha][[\lambda \mathbf{x}_\alpha \mathbf{C}_\beta] \mathbf{A}_\alpha].$
6.  $\mathbf{A}_\alpha\downarrow \supset [\lambda \mathbf{x}_\alpha [\lambda \mathbf{x}_\alpha \mathbf{B}_\beta]] \mathbf{A}_\alpha = \lambda \mathbf{x}_\alpha \mathbf{B}_\beta.$
7.  $\mathbf{A}_\alpha\downarrow \wedge [\text{not-free-in}_{o\epsilon\epsilon} \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\gamma \urcorner \vee \text{not-free-in}_{o\epsilon\epsilon} \ulcorner \mathbf{y}_\beta \urcorner \ulcorner \mathbf{A}_\alpha \urcorner] \supset$   
 $[\lambda \mathbf{x}_\alpha [\lambda \mathbf{y}_\beta \mathbf{B}_\gamma]] \mathbf{A}_\alpha = \lambda \mathbf{y}_\beta [[\lambda \mathbf{x}_\alpha \mathbf{B}_\gamma] \mathbf{A}_\alpha]$  where  $\mathbf{x}_\alpha \neq \mathbf{y}_\beta.$
8.  $[\lambda \mathbf{x}_\alpha [\text{if } \mathbf{B}_o \mathbf{C}_\beta \mathbf{D}_\beta]] \mathbf{A}_\alpha \simeq \text{if } [\lambda \mathbf{x}_\alpha \mathbf{B}_o] \mathbf{A}_\alpha [\lambda \mathbf{x}_\alpha \mathbf{C}_\beta] \mathbf{A}_\alpha [\lambda \mathbf{x}_\alpha \mathbf{D}_\beta] \mathbf{A}_\alpha.$
9.  $\mathbf{A}_\alpha\downarrow \supset [\lambda \mathbf{x}_\alpha \ulcorner \mathbf{B}_\beta \urcorner] \mathbf{A}_\alpha \simeq \ulcorner \mathbf{B}_\beta \urcorner.$
10.  $[\lambda \mathbf{x}_\alpha \mathbf{B}_\beta] \mathbf{x}_\alpha \simeq \mathbf{B}_\beta.$

### Axiom 5 (Tautologous Formulas)

$$\mathbf{A}_o \text{ where } \mathbf{A}_o \text{ is tautologous.}$$

### Axiom 6 (Definedness)

1.  $\mathbf{x}_\alpha\downarrow.$

2.  $\mathbf{c}_\alpha \downarrow$  where  $\mathbf{c}_\alpha$  is a primitive constant.<sup>4</sup>
3.  $\mathbf{A}_{o\beta} \mathbf{B}_\beta \downarrow$ .
4.  $[\mathbf{A}_{\alpha\beta} \uparrow \vee \mathbf{B}_\beta \uparrow] \supset \mathbf{A}_{\alpha\beta} \mathbf{B}_\beta \simeq \perp_\alpha$ .
5.  $[\lambda \mathbf{x}_\alpha \mathbf{B}_\beta] \downarrow$ .
6.  $[\text{if } \mathbf{A}_o \mathbf{B}_o \mathbf{C}_o] \downarrow$ .
7.  $\ulcorner \mathbf{A}_\alpha \urcorner \downarrow$ .
8.  $\llbracket \mathbf{A}_\epsilon \rrbracket_o \downarrow$ .
9.  $\llbracket \ulcorner \mathbf{A}_\alpha \urcorner \rrbracket_\epsilon \downarrow$ .
10.  $\sim [\text{eval-free}_{o\epsilon}^\alpha \mathbf{A}_\epsilon] \supset \llbracket \mathbf{A}_\epsilon \rrbracket_\alpha \simeq \perp_\alpha$ .
11.  $\perp_\alpha \uparrow$  where  $\alpha \neq o$ .

**Axiom 7 (Quasi-Equality)**

1.  $\mathbf{A}_\alpha \simeq \mathbf{A}_\alpha$ .

**Axiom 8 (Definite Description)**

1.  $\exists_1 \mathbf{x}_\alpha \mathbf{A}_o \equiv [\text{I} \mathbf{x}_\alpha \mathbf{A}_o] \downarrow$  where  $\alpha \neq o$ .
2.  $[\exists_1 \mathbf{x}_\alpha \mathbf{A}_o \wedge \text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \text{I} \mathbf{x}_\alpha \mathbf{A}_o \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{A}_o \urcorner = \ulcorner \mathbf{B}_o \urcorner] \supset \mathbf{B}_o$  where  $\alpha \neq o$ .

**Axiom 9 (Ordered Pairs)**

1.  $[\text{pair}_{\langle\alpha\beta\rangle\beta\alpha} \mathbf{A}_\alpha \mathbf{B}_\beta = \text{pair}_{\langle\alpha\beta\rangle\beta\alpha} \mathbf{C}_\alpha \mathbf{D}_\beta] \equiv [\mathbf{A}_\alpha = \mathbf{C}_\alpha \wedge \mathbf{B}_\beta = \mathbf{D}_\beta]$ .
2.  $\mathbf{A}_{\langle\alpha\beta\rangle} \downarrow \supset \exists \mathbf{x}_\alpha \exists \mathbf{y}_\beta [\mathbf{A}_{\langle\alpha\beta\rangle} = \text{pair}_{\langle\alpha\beta\rangle\beta\alpha} \mathbf{x}_\alpha \mathbf{y}_\beta]$ .

**Axiom 10 (Conditionals)**

1.  $[\text{if } T_o \mathbf{B}_\alpha \mathbf{C}_\alpha] \simeq \mathbf{B}_\alpha$ .
2.  $[\text{if } F_o \mathbf{B}_\alpha \mathbf{C}_\alpha] \simeq \mathbf{C}_\alpha$ .
3.  $\llbracket \text{if } \mathbf{A}_o \mathbf{B}_\epsilon \mathbf{C}_\epsilon \rrbracket_\alpha \simeq \text{if } \mathbf{A}_o \llbracket \mathbf{B}_\epsilon \rrbracket_\alpha \llbracket \mathbf{C}_\epsilon \rrbracket_\alpha$ .

**Axiom 11 (Evaluation)**

1.  $\llbracket \ulcorner \mathbf{x}_\alpha \urcorner \rrbracket_\alpha = \mathbf{x}_\alpha$ .
2.  $\llbracket \ulcorner \mathbf{c}_\alpha \urcorner \rrbracket_\alpha = \mathbf{c}_\alpha$  where  $\mathbf{c}_\alpha$  is primitive constant.
3.  $\text{wff}_{o\epsilon}^{\alpha\beta} \mathbf{A}_\epsilon \supset \llbracket \text{app}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon \rrbracket_\alpha \simeq \llbracket \mathbf{A}_\epsilon \rrbracket_{\alpha\beta} \llbracket \mathbf{B}_\epsilon \rrbracket_\beta$ .

---

<sup>4</sup>Notice that, for  $\alpha \neq o$ ,  $\mathbf{c}_\alpha \downarrow$  is false if  $\mathbf{c}_\alpha$  is the defined constant  $\perp_\alpha$ .

4.  $\text{not-free-in}_{o\epsilon\epsilon} \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\epsilon \urcorner \supset \llbracket \text{abs}_{\epsilon\epsilon\epsilon} \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\epsilon \urcorner \rrbracket_{\beta\alpha} \simeq \lambda \mathbf{x}_\alpha \llbracket \mathbf{B}_\epsilon \rrbracket_\beta$ .
5.  $\llbracket \text{cond}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon \mathbf{C}_\epsilon \rrbracket_\alpha \simeq \text{if} \llbracket \mathbf{A}_\epsilon \rrbracket_o \llbracket \mathbf{B}_\epsilon \rrbracket_\alpha \llbracket \mathbf{C}_\epsilon \rrbracket_\alpha$ .
6.  $\llbracket \text{quot}_{\epsilon\epsilon} \mathbf{A}_\epsilon \rrbracket_{\epsilon\downarrow} \supset \llbracket \text{quot}_{\epsilon\epsilon} \mathbf{A}_\epsilon \rrbracket_\epsilon = \mathbf{A}_\epsilon$ .

### Axiom 12 (Specifying Axioms)

$\mathbf{A}_o$  where  $\mathbf{A}_o$  is a specifying axiom in Specifications 1–9.

**Note 20 (Overview of Axioms)** Axioms 1–4 of  $\mathcal{Q}_0^{\text{uqe}}$  correspond to the first four axioms of  $\mathcal{Q}_0$ . Axioms 1 and 2 say essentially the same thing as the first and second axioms of  $\mathcal{Q}_0$  (see the next note). A modification of the third axiom of  $\mathcal{Q}_0$ , Axiom 3 is the axiom of extensionality for partial and total functions. Axiom 4 is the law of beta-reduction for functions that may be partial and arguments that may be undefined. Axiom 4.1 expresses the law of beta-reduction with substitution represented by the logical constant  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$ . Axioms 4.2–9 express the law of beta-reduction using the basic properties of lambda-notation. Axiom 4.10 is an additional property of lambda-notation.

Axiom 5 provides the tautologous formulas that are needed to discharge the definedness conditions and substitution conditions on instances of Axiom 4. Axiom 6 deals with the definedness properties of wffs; the first five parts of Axiom 6 address the three principles of the traditional approach. Axiom 7 states the reflexivity law for quasi-equality. Axioms 8 and 9 state the properties of the logical constants  $\iota_{\alpha(o\alpha)}$  and  $\text{pair}_{(\alpha\beta)\beta\alpha}$ , respectively. Axiom 10 states the properties of conditionals. Axioms 11 states the properties of evaluation. Axiom 12 gives the specifying axioms of the 12 logical constants involving the type  $\epsilon$ .

**Note 21 (Schemas vs. Universal Formulas)** The proof systems  $\mathcal{P}$  and  $\mathcal{P}^u$  are intended to be minimalist axiomatizations of  $\mathcal{Q}_0$  and  $\mathcal{Q}_0^u$ . For instance, in both systems the first three axiom groups are single universal formulas that express three different fundamental ideas. In contrast, the first three axiom groups of  $\mathcal{P}^{\text{uqe}}$  are formula schemas that present all the instances of the three universal formulas. The instances of these universal formulas are obtained in  $\mathcal{P}$  and  $\mathcal{P}^u$  by substitution. Formula schemas are employed in  $\mathcal{P}^{\text{uqe}}$  instead of universal formulas for the sake of convenience and uniformity. In fact, the only axiom presented as a single formula in Axioms 1–12 is Specification 4.29, the principle of induction for type  $\epsilon$ .

**Note 22 (Syntactic Side Conditions)** The syntactic conditions placed on the syntactic variables in the schemas in Axioms 1–12 come in a few simple forms:

1. A syntactic variable  $\mathbf{A}_\alpha$  can be any wff of type  $\alpha$ .
2. A syntactic variable  $\mathbf{x}_\alpha$  can be any variable of type  $\alpha$ .



3. A syntactic variable  $\mathbf{c}_\alpha$  with the condition “ $\mathbf{c}_\alpha$  is a primitive constant” can be any primitive constant of type  $\alpha$ ,
4. A syntactic variable  $\mathbf{A}_\alpha$  with the condition “ $\mathbf{A}_\alpha$  is not a variable” can be any wff of type  $\alpha$  that is not a variable.
5. A syntactic variable  $\mathbf{A}_\alpha$  with the condition “ $\mathbf{A}_\alpha$  is not a primitive constant” can be any wff of type  $\alpha$  that is not a primitive constant.
6. Two variables must be distinct.
7. Two primitive constants must be distinct.
8. Two types must be distinct.

Notice that none of these syntactic side conditions refer to notions concerning free variables and substitution.

## 8.2 Rules of Inference

$\mathcal{Q}_0^{\text{uqe}}$  has just two rules of inference:

**Rule 1 (Quasi-Equality Substitution)** From  $\mathbf{A}_\alpha \simeq \mathbf{B}_\alpha$  and  $\mathbf{C}_o$  infer the result of replacing one occurrence of  $\mathbf{A}_\alpha$  in  $\mathbf{C}_o$  by an occurrence of  $\mathbf{B}_\alpha$ , provided that the occurrence of  $\mathbf{A}_\alpha$  in  $\mathbf{C}_o$  is not within a quotation, not the first argument of a function abstraction, and not the second argument of an evaluation.

**Rule 2 (Modus Ponens)** From  $\mathbf{A}_o$  and  $\mathbf{A}_o \supset \mathbf{B}_o$  infer  $\mathbf{B}_o$ .

**Note 23 (Overview of Rules of Inference)**  $\mathcal{Q}_0^{\text{uqe}}$  has the same two rules of inference as  $\mathcal{Q}_0^{\text{u}}$ . Rule 1 (Quasi-Equality Substitution) corresponds to  $\mathcal{Q}_0$ 's single rule of inference, which is equality substitution. These rules are exactly the same except that the  $\mathcal{Q}_0^{\text{uqe}}$  rule requires only *quasi-equality* ( $\simeq$ ) between the target wff and the substitution wff, while the  $\mathcal{Q}_0$  rule requires *equality* ( $=$ ). Rule 2 (Modus Ponens) is a primitive rule of inference in  $\mathcal{Q}_0^{\text{uqe}}$ , but modus ponens is a derived rule of inference in  $\mathcal{Q}_0$ . Modus ponens must be primitive in  $\mathcal{Q}_0^{\text{uqe}}$  since it is needed to discharge the definedness conditions and substitution conditions on instances of Axiom 4, the law of beta-reduction.

## 8.3 Proofs

Let  $\mathbf{A}_o$  be a formula and  $\mathcal{H}$  be a set of sentences (i.e., semantically closed formulas) of  $\mathcal{Q}_0^{\text{uqe}}$ . A *proof of  $\mathbf{A}_o$  from  $\mathcal{H}$  in  $\mathcal{P}^{\text{uqe}}$*  is a finite sequence of wffs, ending with  $\mathbf{A}_o$ , such that each member in the sequence is an axiom of  $\mathcal{P}^{\text{uqe}}$ , a member of  $\mathcal{H}$ , or is inferred from preceding members in the sequence by a rule of inference of  $\mathcal{P}^{\text{uqe}}$ . We write  $\mathcal{H} \vdash \mathbf{A}_o$  to mean there is a proof of  $\mathbf{A}_o$  from  $\mathcal{H}$  in  $\mathcal{P}^{\text{uqe}}$ .  $\vdash \mathbf{A}_o$  is written instead of  $\emptyset \vdash \mathbf{A}_o$ .  $\mathbf{A}_o$  is a *theorem* of  $\mathcal{P}^{\text{uqe}}$  if  $\vdash \mathbf{A}_o$ .

Now let  $\mathcal{H}$  be a set of syntactically closed evaluation-free formulas of  $\mathcal{Q}_0^{\text{uqe}}$ . (Recall that a syntactically closed evaluation-free formula is also semantically closed by Lemma 7.2.4.) An *evaluation-free proof of  $\mathbf{A}_o$  from  $\mathcal{H}$  in  $\mathcal{P}^{\text{uqe}}$*  is a proof of  $\mathbf{A}_o$  from  $\mathcal{H}$  that is a sequence of evaluation-free wffs $_o$ . We write  $\mathcal{H} \vdash^{\text{ef}} \mathbf{A}_o$  to mean there is an evaluation-free proof of  $\mathbf{A}_o$  from  $\mathcal{H}$  in  $\mathcal{P}^{\text{uqe}}$ . Obviously,  $\mathcal{H} \vdash^{\text{ef}} \mathbf{A}_o$  implies  $\mathcal{H} \vdash \mathbf{A}_o$ .  $\vdash^{\text{ef}} \mathbf{A}_o$  is written instead of  $\emptyset \vdash^{\text{ef}} \mathbf{A}_o$ .

$\mathcal{H}$  is *consistent* in  $\mathcal{P}^{\text{uqe}}$  if there is no proof of  $F_o$  from  $\mathcal{H}$  in  $\mathcal{P}^{\text{uqe}}$ .

**Note 24 (Proof from Hypotheses)** Andrews employs in [2] a more complicated notion of a “proof from hypotheses” in which a hypothesis is not required to be semantically or syntactically closed. We have chosen to use the simpler notion since it is difficult to define Andrews’ notion in the presence of evaluations and we can manage well enough in this paper with having only semantically or syntactically closed hypotheses.

## 9 Soundness

$\mathcal{P}^{\text{uqe}}$  is *sound* for  $\mathcal{Q}_0^{\text{uqe}}$  if  $\mathcal{H} \vdash \mathbf{A}_o$  implies  $\mathcal{H} \models_n \mathbf{A}_o$  whenever  $\mathbf{A}_o$  is a formula and  $\mathcal{H}$  is a set of sentences of  $\mathcal{Q}_0^{\text{uqe}}$ . We will prove that the proof system  $\mathcal{P}^{\text{uqe}}$  is sound for  $\mathcal{Q}_0^{\text{uqe}}$  by showing that its axioms are valid in every normal general model for  $\mathcal{Q}_0^{\text{uqe}}$  and its rules of inference preserve validity in every normal general model for  $\mathcal{Q}_0^{\text{uqe}}$ .

### 9.1 Axioms and Rules of Inference

**Lemma 9.1.1** *Each axiom of  $\mathcal{P}^{\text{uqe}}$  is valid in every normal general model for  $\mathcal{Q}_0^{\text{uqe}}$ .*

**Proof** Let  $\mathcal{M} = \langle \{\mathcal{D}_\alpha \mid \alpha \in \mathcal{T}\}, \mathcal{J} \rangle$  be a normal general model for  $\mathcal{Q}_0^{\text{uqe}}$  and  $\varphi \in \text{assign}(\mathcal{M})$ . There are 16 cases, one for each group of axioms.

**Axiom 1** The proof is similar to the proof of 5402 for Axiom 1 in [2, p. 241] when  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{G}_{oo})$  is defined. The proof is straightforward when  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{G}_{oo})$  is undefined.

**Axiom 2** The proof is similar to the proof of 5402 for Axiom 2 in [2, p. 242] when  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{H}_{o\alpha})$  is defined. The proof is straightforward when  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{H}_{o\alpha})$  is undefined.

**Axiom 3** The proof is similar to the proof of 5402 for Axiom 3 in [2, p. 242].

**Axiom 4**

**Axiom 4.1** Each instance of Axiom 4.1 is valid in  $\mathcal{M}$  by Theorem 7.5.2.

**Axiom 4.2** We must show

$$(a) \mathcal{V}_\varphi^{\mathcal{M}}([\lambda \mathbf{x}_\alpha \mathbf{x}_\alpha] \mathbf{A}_\alpha) \simeq \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)$$

to prove Axiom 4.2 is valid in  $\mathcal{M}$ . If  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)$  is undefined, then clearly (a) is true. So assume (b)  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)$  is defined. Then

$$\mathcal{V}_\varphi^{\mathcal{M}}([\lambda \mathbf{x}_\alpha \mathbf{x}_\alpha] \mathbf{A}_\alpha) \quad (1)$$

$$\simeq \mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)]}^{\mathcal{M}}(\mathbf{x}_\alpha) \quad (2)$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha) \quad (3)$$

(2) is by (b) and the semantics of function application and function abstraction, and (3) is by the semantics of variables.

**Axiom 4.3** Assume (a)  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)$  is defined and (b)  $\mathbf{x}_\alpha \neq \mathbf{y}_\beta$ . We must show

$$\mathcal{V}_\varphi^{\mathcal{M}}([\lambda \mathbf{x}_\alpha \mathbf{y}_\beta] \mathbf{A}_\alpha) \simeq \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{y}_\beta)$$

to prove Axiom 4.3 is valid in  $\mathcal{M}$ . Then

$$\mathcal{V}_\varphi^{\mathcal{M}}([\lambda \mathbf{x}_\alpha \mathbf{y}_\beta] \mathbf{A}_\alpha) \quad (1)$$

$$\simeq \mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)]}^{\mathcal{M}}(\mathbf{y}_\beta) \quad (2)$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{y}_\beta). \quad (3)$$

(2) is by (a) and the semantics of function application and function abstraction, and (3) is by (b) and the semantics of variables.

**Axiom 4.4** Similar to Axiom 4.3.

**Axiom 4.5** We must show

$$(a) \mathcal{V}_\varphi^{\mathcal{M}}([\lambda \mathbf{x}_\alpha [\mathbf{B}_{\alpha\beta} \mathbf{C}_\beta]] \mathbf{A}_\alpha) \simeq \mathcal{V}_\varphi^{\mathcal{M}}([\lambda \mathbf{x}_\alpha \mathbf{B}_{\alpha\beta}] \mathbf{A}_\alpha) [[\lambda \mathbf{x}_\alpha \mathbf{C}_\beta] \mathbf{A}_\alpha]$$

to prove Axiom 4.5 is valid in  $\mathcal{M}$ . If  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)$  is undefined, then clearly (a) is true. So assume (b)  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)$  is defined. Then

$$\mathcal{V}_\varphi^{\mathcal{M}}([\lambda \mathbf{x}_\alpha [\mathbf{B}_{\alpha\beta} \mathbf{C}_\beta]] \mathbf{A}_\alpha) \quad (1)$$

$$\simeq \mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)]}^{\mathcal{M}}(\mathbf{B}_{\alpha\beta} \mathbf{C}_\beta) \quad (2)$$

$$\simeq \mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)]}^{\mathcal{M}}(\mathbf{B}_{\alpha\beta}) (\mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)]}^{\mathcal{M}}(\mathbf{C}_\beta)) \quad (3)$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}([\lambda \mathbf{x}_\alpha \mathbf{B}_{\alpha\beta}] \mathbf{A}_\alpha) (\mathcal{V}_\varphi^{\mathcal{M}}([\lambda \mathbf{x}_\alpha \mathbf{C}_\beta] \mathbf{A}_\alpha)) \quad (4)$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}([\lambda \mathbf{x}_\alpha \mathbf{B}_{\alpha\beta}] \mathbf{A}_\alpha) [[\lambda \mathbf{x}_\alpha \mathbf{C}_\beta] \mathbf{A}_\alpha]. \quad (5)$$

(2) and (4) are by (b) and the semantics of function application and function abstraction, and (3) and (5) are by the semantics of function application.

**Axiom 4.6** Assume (a)  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)$  is defined. We must show

$$\mathcal{V}_\varphi^{\mathcal{M}}([\lambda \mathbf{x}_\alpha [\lambda \mathbf{x}_\alpha \mathbf{B}_\beta]] \mathbf{A}_\alpha)(d) \simeq \mathcal{V}_\varphi^{\mathcal{M}}(\lambda \mathbf{x}_\alpha \mathbf{B}_\beta)(d),$$

where  $d \in \mathcal{D}_\alpha$ , to prove Axiom 4.6 is valid in  $\mathcal{M}$ .

$$\mathcal{V}_\varphi^{\mathcal{M}}([\lambda \mathbf{x}_\alpha [\lambda \mathbf{x}_\alpha \mathbf{B}_\beta]] \mathbf{A}_\alpha)(d) \tag{1}$$

$$\simeq \mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)]}^{\mathcal{M}}(\lambda \mathbf{x}_\alpha \mathbf{B}_\beta)(d) \tag{2}$$

$$\simeq \mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)][\mathbf{x}_\alpha \mapsto d]}^{\mathcal{M}}(\mathbf{B}_\beta) \tag{3}$$

$$\simeq \mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto d]}^{\mathcal{M}}(\mathbf{B}_\beta) \tag{4}$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\lambda \mathbf{x}_\alpha \mathbf{B}_\beta)(d). \tag{5}$$

(2) is by (a) and the semantics of function application and function abstraction; (3) and (5) are by the semantics of function abstraction; and (4) is by

$$\varphi[\mathbf{x}_\alpha \mapsto \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)][\mathbf{x}_\alpha \mapsto d] = \varphi[\mathbf{x}_\alpha \mapsto d].$$

**Axiom 4.7** Assume (a)  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)$  is defined, (b)  $\mathbf{x}_\alpha \neq \mathbf{y}_\beta$ , and

$$(c) \mathcal{M} \models \text{not-free-in}_{o \in \epsilon} \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\gamma \urcorner \text{ or}$$

$$\mathcal{M} \models \text{not-free-in}_{o \in \epsilon} \ulcorner \mathbf{y}_\beta \urcorner \ulcorner \mathbf{A}_\alpha \urcorner.$$

We must show

$$\mathcal{V}_\varphi^{\mathcal{M}}([\lambda \mathbf{x}_\alpha [\lambda \mathbf{y}_\beta \mathbf{B}_\gamma]] \mathbf{A}_\alpha)(d) \simeq \mathcal{V}_\varphi^{\mathcal{M}}(\lambda \mathbf{y}_\beta [[\lambda \mathbf{x}_\alpha \mathbf{B}_\gamma] \mathbf{A}_\alpha])(d),$$

where  $d \in \mathcal{D}_\beta$ , to prove Axiom 4.7 is valid in  $\mathcal{M}$ .

$$\mathcal{V}_\varphi^{\mathcal{M}}([\lambda \mathbf{x}_\alpha [\lambda \mathbf{y}_\beta \mathbf{B}_\gamma]] \mathbf{A}_\alpha)(d) \tag{1}$$

$$\simeq \mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)]}^{\mathcal{M}}(\lambda \mathbf{y}_\beta \mathbf{B}_\gamma)(d) \tag{2}$$

$$\simeq \mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)][\mathbf{y}_\beta \mapsto d]}^{\mathcal{M}}(\mathbf{B}_\gamma) \tag{3}$$

$$\simeq \mathcal{V}_{\varphi[\mathbf{y}_\beta \mapsto d][\mathbf{x}_\alpha \mapsto \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)]}^{\mathcal{M}}(\mathbf{B}_\gamma) \tag{4}$$

$$\simeq \mathcal{V}_{\varphi[\mathbf{y}_\beta \mapsto d][\mathbf{x}_\alpha \mapsto \mathcal{V}_{\varphi[\mathbf{y}_\beta \mapsto d]}^{\mathcal{M}}(\mathbf{A}_\alpha)]}^{\mathcal{M}}(\mathbf{B}_\gamma) \tag{5}$$

$$\simeq \mathcal{V}_{\varphi[\mathbf{y}_\beta \mapsto d]}^{\mathcal{M}}([\lambda \mathbf{x}_\alpha \mathbf{B}_\gamma] \mathbf{A}_\alpha) \tag{6}$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\lambda \mathbf{y}_\beta [[\lambda \mathbf{x}_\alpha \mathbf{B}_\gamma] \mathbf{A}_\alpha])(d) \tag{7}$$

(2) and (6) are by (a) and the semantics of function application and function abstraction; (3) and (7) are by the semantics of function abstraction; (4) is by (b); and (5) is by (c) and part 2 of Lemma 7.3.1.

**Axiom 4.8** Similar to Axiom 4.5.

**Axiom 4.9** Similar to Axiom 4.3.

**Axiom 4.10** We must show

$$\mathcal{V}_\varphi^{\mathcal{M}}([\lambda \mathbf{x}_\alpha \mathbf{B}_\beta] \mathbf{x}_\alpha) \simeq \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_\beta)$$

to prove Axiom 4.10 is valid in  $\mathcal{M}$ .

$$\mathcal{V}_\varphi^{\mathcal{M}}([\lambda \mathbf{x}_\alpha \mathbf{B}_\beta] \mathbf{x}_\alpha) \tag{1}$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}([\lambda \mathbf{x}_\alpha \mapsto \varphi(\mathbf{x}_\alpha)](\mathbf{B}_\beta)) \tag{2}$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_\beta) \tag{3}$$

(2) is by the semantics of function application, function abstraction, variables; and (3) is by  $\varphi = \varphi[\mathbf{x}_\alpha \mapsto \varphi(\mathbf{x}_\alpha)]$ .

**Axiom 5** The propositional constants  $T_o$  and  $F_o$  and the propositional connectives  $\wedge_{ooo}$ ,  $\vee_{ooo}$ , and  $\supset_{ooo}$  have their usual meanings in a general model. Hence any tautologous formula is valid in  $\mathcal{M}$ .

**Axiom 6**  $\mathcal{M} \models \mathbf{A}_\alpha \downarrow$  iff  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)$  is defined for all  $\varphi \in \text{assign}(\mathcal{M})$ . Hence Axioms 6.1, 6.2, 6.3, 6.4, 6.5, 6.6, 6.7, and 6.8 are valid in  $\mathcal{M}$  by conditions 1, 2, 3, 3, 4, 5, 6, and 7 in the definition of a general model. Axiom 6.9 is valid in  $\mathcal{M}$  by the fact that quotations are evaluation-free and conditions 6 and 7 in definition of a general model. Axiom 6.10 is valid in  $\mathcal{M}$  by Proposition 7.2.1 and condition 7 in the definition of a general model. Axiom 6.11 is valid in  $\mathcal{M}$  since  $\mathcal{J}(\iota_{\alpha(o\alpha)})$  is a unique member selector on  $\mathcal{D}_\alpha$  and  $\lambda x_\alpha [x_\alpha \neq x_\alpha]$  represents the empty set.

**Axiom 7** Clearly,  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha \simeq \mathbf{A}_\alpha) = \top$  iff  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha) \simeq \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha)$ , which is always true. Hence  $\mathcal{M} \models \mathbf{A}_\alpha \simeq \mathbf{A}_\alpha$ .

**Axiom 8**

**Axiom 8.1** Axiom 8.1 is valid in  $\mathcal{M}$  since  $\mathcal{J}(\iota_{\alpha(o\alpha)})$  is a unique member selector on  $\mathcal{D}_\alpha$ .

**Axiom 8.2** Assume (a)  $\mathcal{V}_\varphi^{\mathcal{M}}(\exists_1 \mathbf{x}_\alpha \mathbf{A}_o) = \top$  and

$$\mathcal{V}_\varphi^{\mathcal{M}}(\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{I} \mathbf{x}_\alpha \mathbf{A}_o \urcorner \lrcorner \mathbf{x}_\alpha \lrcorner \ulcorner \mathbf{A}_o \urcorner = \ulcorner \mathbf{B}_o \urcorner) = \top.$$

We must show  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_o) = \top$  to prove that Axiom 8.2 is valid in  $\mathcal{M}$ . Axiom 8.1 and (a) implies  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{I} \mathbf{x}_\alpha \mathbf{A}_o)$  is defined. (a) and the fact that  $\mathcal{J}(\iota_{\alpha(o\alpha)})$  is a unique member selector on  $\mathcal{D}_\alpha$  implies

$$\mathcal{V}_\varphi^{\mathcal{M}}([\lambda \mathbf{x}_\alpha \mathbf{A}_o][\mathbf{I} \mathbf{x}_\alpha \mathbf{A}_o]) = \top.$$

Then  $\mathcal{V}_\varphi^{\mathcal{M}}([\lambda \mathbf{x}_\alpha \mathbf{A}_o][\mathbf{I} \mathbf{x}_\alpha \mathbf{A}_o] = \mathbf{B}_o)$  by the proof for Axiom 4. Thus  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_o) = \top$ .

**Axiom 9** Axiom 9.1 is valid in  $\mathcal{M}$  since  $\mathcal{J}(\text{pair}_{\langle\alpha\beta\rangle\beta\alpha})$  is a pairing function on  $\mathcal{D}_\alpha$  and  $\mathcal{D}_\beta$ . Axiom 9.2 is valid in  $\mathcal{M}$  since every  $p \in \mathcal{D}_{\langle\alpha\beta\rangle}$  is a pair  $\langle a, b \rangle$  where  $a \in \mathcal{D}_\alpha$  and  $b \in \mathcal{D}_\beta$  and  $\mathcal{J}(\text{pair}_{\langle\alpha\beta\rangle\beta\alpha})$  is a pairing function on  $\mathcal{D}_\alpha$  and  $\mathcal{D}_\beta$ .

**Axiom 10** Axioms 10.1 and 10.2 are valid in  $\mathcal{M}$  by condition 5 in the definition of a general model.  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_o) = \top$  implies  $\mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \text{if } \mathbf{A}_o \mathbf{B}_\epsilon \mathbf{C}_\epsilon \rrbracket_\alpha) \simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \mathbf{B}_\epsilon \rrbracket_\alpha)$  and  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_o) = \text{F}$  implies  $\mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \text{if } \mathbf{A}_o \mathbf{B}_\epsilon \mathbf{C}_\epsilon \rrbracket_\alpha) \simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \mathbf{C}_\epsilon \rrbracket_\alpha)$  by conditions 5 and 7 in the definition of a general model. Hence Axiom 10.3 is valid in  $\mathcal{M}$ .

**Axiom 11**

**Axioms 11.1 and 11.2** Immediate by condition 7 in the definition of a general model since variables and primitive constants are evaluation-free.

**Axiom 11.3** Assume  $\mathcal{V}_\varphi^{\mathcal{M}}(\text{wff}_{o\epsilon}^{\alpha\beta} \mathbf{A}_\epsilon) = \top$ . This implies (a)  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\epsilon) = \mathcal{E}(\mathbf{C}_{\alpha\beta})$  for some  $\mathbf{C}_{\alpha\beta}$ . We must show  $X \simeq Y$  where  $X$  is

$$\mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \text{app}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon \rrbracket_\alpha)$$

and  $Y$  is

$$\mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \mathbf{A}_\epsilon \rrbracket_{\alpha\beta} \llbracket \mathbf{B}_\epsilon \rrbracket_\beta).$$

First, assume (b)  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_\epsilon) = \mathcal{E}(\mathbf{D}_\beta)$  for some  $\mathbf{D}_\beta$ . If (c)  $\mathbf{C}_{\alpha\beta} \mathbf{D}_\beta$  is evaluation-free, then

$$\mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \text{app}_{\epsilon\epsilon\epsilon} \mathbf{A}_\epsilon \mathbf{B}_\epsilon \rrbracket_\alpha) \tag{1}$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \text{app}_{\epsilon\epsilon\epsilon} \mathcal{E}(\mathbf{C}_{\alpha\beta}) \mathcal{E}(\mathbf{D}_\beta) \rrbracket_\alpha) \tag{2}$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \mathcal{E}(\mathbf{C}_{\alpha\beta} \mathbf{D}_\beta) \rrbracket_\alpha) \tag{3}$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{C}_{\alpha\beta} \mathbf{D}_\beta) \tag{4}$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{C}_{\alpha\beta})(\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\beta)) \tag{5}$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \mathcal{E}(\mathbf{C}_{\alpha\beta}) \rrbracket_{\alpha\beta})(\mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \mathcal{E}(\mathbf{D}_\beta) \rrbracket_\beta)) \tag{6}$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \mathbf{A}_\epsilon \rrbracket_{\alpha\beta})(\mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \mathbf{B}_\epsilon \rrbracket_\beta)) \tag{7}$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \mathbf{A}_\epsilon \rrbracket_{\alpha\beta} \llbracket \mathbf{B}_\epsilon \rrbracket_\beta). \tag{8}$$

(2) and (7) are by (a), (b), and Proposition 6.2.1; (3) is by the definition of  $\mathcal{E}$ ; (4) and (6) are by (c) and the semantics of evaluation; and (5) and (8) are by the semantics of function application. Hence  $X \simeq Y$ . If  $\mathbf{C}_{\alpha\beta} \mathbf{D}_\beta$  is not evaluation-free, then  $\mathbf{C}_{\alpha\beta}$  or  $\mathbf{D}_\beta$  is not evaluation-free. Then  $X$  and  $Y$  are both undefined by the semantics of evaluation and the beginning and end of the derivation above. Hence  $X \simeq Y$ .

Second, assume  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_\epsilon) = \mathcal{E}(\mathbf{D}_\gamma)$  with  $\gamma \neq \beta$ . Then  $X$  is undefined by Specifications 6.4 and 6.5, the semantics of evaluation, and the beginning of the derivation above, and  $Y$  is undefined by the semantics of evaluation

and function application and the end of the derivation above. Hence  $X \simeq Y$ .

Third, assume  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_\epsilon) = d$  where  $d$  is a nonstandard construction. Then  $X$  is undefined by Lemmas 6.3.2 and 6.3.3, and  $Y$  is undefined by Lemma 6.3.2 and the semantics of function application. Hence  $X \simeq Y$  in this case, and therefore, in every case.

**Axiom 11.4** Let

$$(a) \mathcal{V}_\varphi^{\mathcal{M}}(\text{not-free-in}_{oe\epsilon} \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\epsilon \urcorner) = \top$$

and  $d \in \mathcal{D}_\alpha$ . It suffices to show  $X(d) \simeq Y(d)$  where  $X$  is

$$\mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \text{abs}_{\epsilon\epsilon\epsilon} \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\epsilon \urcorner \rrbracket_{\beta\alpha})$$

and  $Y$  is

$$\mathcal{V}_\varphi^{\mathcal{M}}(\lambda \mathbf{x}_\alpha \llbracket \mathbf{B}_\epsilon \rrbracket_\beta).$$

First, assume (b)  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_\epsilon) = \mathcal{E}(\mathbf{C}_\beta)$  for some  $\mathbf{C}_\beta$ . This implies (c)  $\mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto d]}^{\mathcal{M}}(\mathbf{B}_\epsilon) = \mathcal{E}(\mathbf{C}_\beta)$  by (a) and part 2 of Lemma 7.3.1. If (d)  $\mathbf{C}_\beta$  is evaluation-free, then

$$\mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \text{abs}_{\epsilon\epsilon\epsilon} \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\epsilon \urcorner \rrbracket_{\beta\alpha})(d) \tag{1}$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \text{abs}_{\epsilon\epsilon\epsilon} \ulcorner \mathbf{x}_\alpha \urcorner \mathcal{E}(\mathbf{C}_\beta) \rrbracket_{\beta\alpha})(d) \tag{2}$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \mathcal{E}(\lambda \mathbf{x}_\alpha \mathbf{C}_\beta) \rrbracket_{\beta\alpha})(d) \tag{3}$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\lambda \mathbf{x}_\alpha \mathbf{C}_\beta)(d) \tag{4}$$

$$\simeq \mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto d]}^{\mathcal{M}}(\mathbf{C}_\beta) \tag{5}$$

$$\simeq \mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto d]}^{\mathcal{M}}(\llbracket \mathcal{E}(\mathbf{C}_\beta) \rrbracket_\beta) \tag{6}$$

$$\simeq \mathcal{V}_{\varphi[\mathbf{x}_\alpha \mapsto d]}^{\mathcal{M}}(\llbracket \mathbf{B}_\epsilon \rrbracket_\beta) \tag{7}$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\lambda \mathbf{x}_\alpha \llbracket \mathbf{B}_\epsilon \rrbracket_\beta)(d). \tag{8}$$

(2) is by (b) and Proposition 6.2.1; (3) is by the definition of  $\mathcal{E}$ ; (4) and (6) are by the semantics of evaluation and (d); (5) and (8) are by the semantics of function abstraction; and (7) is by (c) and Proposition 6.2.1. Hence  $X(d) \simeq Y(d)$ . If  $\mathbf{C}_\beta$  is not evaluation-free, then  $X(d)$  and  $Y(d)$  are both undefined by the semantics of evaluation and the beginning and end of the derivation above. Hence  $X(d) \simeq Y(d)$ .

Second, assume  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_\epsilon) = \mathcal{E}(\mathbf{C}_\gamma)$  for some  $\mathbf{C}_\gamma$  where  $\gamma \neq \beta$ . Then  $X(d)$  is undefined by the semantics of evaluation and the beginning of the derivation above, and  $Y(d)$  is undefined by the semantics of evaluation and the end of the derivation above. Hence  $X(d) \simeq Y(d)$ .

Third, assume  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_\epsilon)$  is a nonstandard construction. Then  $X(d)$  is undefined by Lemmas 6.3.2 and 6.3.3, and  $Y(d)$  is undefined by Lemma 6.3.2. Hence  $X(d) \simeq Y(d)$  in this case, and therefore, in every case.

**Axiom 11.5** Similar to Axiom 11.3.

**Axiom 11.6** First, assume  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\epsilon) = \mathcal{E}(\mathbf{B}_\alpha)$  for some  $\mathbf{B}_\alpha$ . Then

$$\mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \text{quot}_{\epsilon\epsilon} \mathbf{A}_\epsilon \rrbracket_\epsilon). \quad (1)$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \text{quot}_{\epsilon\epsilon} \mathcal{E}(\mathbf{B}_\alpha) \rrbracket_\epsilon). \quad (2)$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \mathcal{E}(\ulcorner \mathbf{B}_\alpha \urcorner) \rrbracket_\epsilon). \quad (3)$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\ulcorner \mathbf{B}_\alpha \urcorner). \quad (4)$$

$$\simeq \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\epsilon). \quad (5)$$

(2) is by Proposition 6.2.1; (3) is by the definition of  $\mathcal{E}$ ; (4) by the semantics of evaluation and the fact that quotations are evaluation-free; and (5) is by Specification 1. Hence  $\mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \text{quot}_{\epsilon\epsilon} \mathbf{A}_\epsilon \rrbracket_\epsilon) = \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\epsilon)$  since  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\epsilon)$  is defined.

Second, assume  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\epsilon) \neq \mathcal{E}(\mathbf{B}_\alpha)$  for all  $\mathbf{B}_\alpha$ . Then  $\mathcal{V}_\varphi^{\mathcal{M}}(\text{quot}_{\epsilon\epsilon} \mathbf{A}_\epsilon) \neq \mathcal{E}(\mathbf{A}_\alpha)$  for all  $\mathbf{B}_\alpha$  by Lemma 6.3.3. Hence  $\mathcal{V}_\varphi^{\mathcal{M}}(\llbracket \text{quot}_{\epsilon\epsilon} \mathbf{A}_\epsilon \rrbracket_\epsilon)$  is undefined by Lemma 6.3.2. Therefore, Axiom 11.6 is valid in  $\mathcal{M}$  in both cases.

**Axiom 12** Each axiom of this group is a specifying axiom and thus is valid in  $\mathcal{M}$  since  $\mathcal{M}$  is normal.  $\square$

**Lemma 9.1.2** *Each rule of inference of  $\mathcal{P}^{\text{uqe}}$  preserves validity in every normal general model for  $\mathcal{Q}_0^{\text{uqe}}$ .*

**Proof** Let  $\mathcal{M}$  be a normal general model for  $\mathcal{Q}_0^{\text{uqe}}$ . We must show that Rules 1 and 2 preserve validity in  $\mathcal{M}$ .

**Rule 1** Suppose  $\mathbf{C}_o$  and  $\mathbf{C}'_o$  are wffs such that  $\mathbf{C}'_o$  is the result of replacing one occurrence of  $\mathbf{A}_\alpha$  in  $\mathbf{C}_o$  by an occurrence of  $\mathbf{B}_\alpha$ , provided that the occurrence of  $\mathbf{A}_\alpha$  in  $\mathbf{C}_o$  is not within a quotation, not the first argument of a function abstraction, and not the second argument of an evaluation. Then it easily follows that  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha) \simeq \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_\alpha)$  for all  $\varphi \in \text{assign}(\mathcal{M})$  implies  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{C}_o) = \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{C}'_o)$  for all  $\varphi \in \text{assign}(\mathcal{M})$  by induction on the size of  $\mathbf{C}_o$ . Hence  $\mathcal{M} \models \mathbf{A}_\alpha \simeq \mathbf{B}_\alpha$  implies  $\mathcal{M} \models \mathbf{C}_o = \mathbf{C}'_o$  since  $\mathcal{M} \models \mathbf{A}_\alpha \simeq \mathbf{B}_\alpha$  implies  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_\alpha) \simeq \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_\alpha)$  for all  $\varphi \in \text{assign}(\mathcal{M})$ . Therefore, Rule 1 preserves validity in  $\mathcal{M}$ .

**Rule 2** Since  $\supset_{ooo}$  has its usual meaning in a general model, Rule 2 obviously preserves validity in  $\mathcal{M}$ .  $\square$

## 9.2 Soundness and Consistency Theorems

**Theorem 9.2.1 (Soundness Theorem)**  *$\mathcal{P}^{\text{uqe}}$  is sound for  $\mathcal{Q}_0^{\text{uqe}}$ .*

**Proof** Assume  $\mathcal{H} \vdash \mathbf{A}_o$  and  $\mathcal{M} \models \mathcal{H}$  where  $\mathbf{A}_o$  is a formula of  $\mathcal{Q}_0^{\text{uqe}}$ ,  $\mathcal{H}$  is a set of sentences of  $\mathcal{Q}_0^{\text{uqe}}$ , and  $\mathcal{M}$  is a normal general model for  $\mathcal{Q}_0^{\text{uqe}}$ . We must show that  $\mathcal{M} \models \mathbf{A}_o$ . By Lemma 9.1.1, each axiom of  $\mathcal{P}^{\text{uqe}}$  is valid in  $\mathcal{M}$ , and by Lemma 9.1.2, each rule of inference of  $\mathcal{P}^{\text{uqe}}$  preserve validity in  $\mathcal{M}$ . Therefore,  $\mathcal{H} \vdash \mathbf{A}_o$  implies  $\mathcal{M} \models \mathbf{A}_o$ .  $\square$



**Theorem 9.2.2 (Consistency Theorem)** *Let  $\mathcal{H}$  be a set of sentences of  $\mathcal{Q}_0^{\text{uqe}}$ . If  $\mathcal{H}$  has a normal general model, then  $\mathcal{H}$  is consistent in  $\mathcal{P}^{\text{uqe}}$ .*

**Proof** Let  $\mathcal{M}$  be a normal general model for  $\mathcal{H}$ . Assume that  $\mathcal{H}$  is inconsistent in  $\mathcal{P}^{\text{uqe}}$ , i.e., that  $\mathcal{H} \vdash F_o$ . Then, by the Soundness Theorem,  $\mathcal{H} \models_n F_o$  and hence  $\mathcal{M} \models F_o$ . This means that  $\mathcal{V}_\varphi^{\mathcal{M}}(F_o) = \top$  and thus  $\mathcal{V}_\varphi^{\mathcal{M}}(F_o) \neq \text{F}$  (for any assignment  $\varphi$ ), which contradicts the definition of a general model.  $\square$

## 10 Some Metatheorems

We will prove several metatheorems of  $\mathcal{Q}_0^{\text{uqe}}$ . Most of them will be metatheorems that we need in order to prove the evaluation-free completeness of  $\mathcal{Q}_0^{\text{uqe}}$  in section 11 and the results in section 12.

### 10.1 Analogs to Metatheorems of $\mathcal{Q}_0$

Most of the metatheorems we prove in this subsection are analogs of the metatheorems of  $\mathcal{Q}_0$  proven in section 52 of [2]. There will be two versions for many of them, the first restricted to evaluation-free proofs and the second unrestricted. In this subsection, let  $\mathcal{H}^{\text{ef}}$  be a set of syntactically closed evaluation-free formulas of  $\mathcal{Q}_0^{\text{uqe}}$  and  $\mathcal{H}$  be a set of sentences of  $\mathcal{Q}_0^{\text{uqe}}$ .

**Proposition 10.1.1 (Analog of 5200 in [2])**

1.  $\vdash^{\text{ef}} \mathbf{A}_\alpha \simeq \mathbf{A}_\alpha$  where  $\mathbf{A}_\alpha$  is evaluation-free.
2.  $\vdash \mathbf{A}_\alpha \simeq \mathbf{A}_\alpha$ .

**Proof** By Axiom 7 for both parts.  $\square$

**Theorem 10.1.2 (Tautology Theorem: Analog of 5234)**

1. Let  $\mathbf{A}_o^1, \dots, \mathbf{A}_o^n, \mathbf{B}_o$  be evaluation-free. If  $\mathcal{H}^{\text{ef}} \vdash^{\text{ef}} \mathbf{A}_o^1, \dots, \mathcal{H}^{\text{ef}} \vdash^{\text{ef}} \mathbf{A}_o^n$  and  $[\mathbf{A}_o^1 \wedge \dots \wedge \mathbf{A}_o^n] \supset \mathbf{B}_o$  is tautologous for  $n \geq 1$ , then  $\mathcal{H}^{\text{ef}} \vdash^{\text{ef}} \mathbf{B}_o$ . Also, if  $\mathbf{B}_o$  is tautologous, then  $\mathcal{H}^{\text{ef}} \vdash^{\text{ef}} \mathbf{B}_o$ .
2. If  $\mathcal{H} \vdash \mathbf{A}_o^1, \dots, \mathcal{H} \vdash \mathbf{A}_o^n$  and  $[\mathbf{A}_o^1 \wedge \dots \wedge \mathbf{A}_o^n] \supset \mathbf{B}_o$  is tautologous for  $n \geq 1$ , then  $\mathcal{H} \vdash \mathbf{B}_o$ . Also, if  $\mathbf{B}_o$  is tautologous, then  $\mathcal{H} \vdash \mathbf{B}_o$ .

**Proof** Follows from Axiom 5 (Tautologous Formulas) and Rule 2 (Modus Ponens) for both parts.  $\square$

**Lemma 10.1.3**

1.  $\vdash^{\text{ef}} [\mathbf{A}_\alpha = \mathbf{B}_\alpha] \supset [\mathbf{A}_\alpha \simeq \mathbf{B}_\alpha]$  where  $\mathbf{A}_\alpha$  and  $\mathbf{B}_\alpha$  are evaluation-free.
2.  $\vdash [\mathbf{A}_\alpha = \mathbf{B}_\alpha] \supset [\mathbf{A}_\alpha \simeq \mathbf{B}_\alpha]$ .

**Proof** Follows from the definition of  $\simeq$  and the Tautology Theorem for both parts.  $\square$

**Lemma 10.1.4**

1. If  $\mathcal{H}^{\text{ef}} \vdash^{\text{ef}} \mathbf{A}_\alpha \downarrow$  or  $\mathcal{H}^{\text{ef}} \vdash^{\text{ef}} \mathbf{B}_\alpha \downarrow$ , then  $\mathcal{H}^{\text{ef}} \vdash^{\text{ef}} \mathbf{A}_\alpha \simeq \mathbf{B}_\alpha$  implies  $\mathcal{H}^{\text{ef}} \vdash^{\text{ef}} \mathbf{A}_\alpha = \mathbf{B}_\alpha$  where  $\mathbf{A}_\alpha$  and  $\mathbf{B}_\alpha$  are evaluation-free.
2. If  $\mathcal{H} \vdash \mathbf{A}_\alpha \downarrow$  or  $\mathcal{H} \vdash \mathbf{B}_\alpha \downarrow$ , then  $\mathcal{H} \vdash \mathbf{A}_\alpha \simeq \mathbf{B}_\alpha$  implies  $\mathcal{H} \vdash \mathbf{A}_\alpha = \mathbf{B}_\alpha$ .

**Proof** Follows from the definition of  $\simeq$  and the Tautology Theorem for both parts.  $\square$

**Corollary 10.1.5**  $\vdash^{\text{ef}} T_o$ .

**Proof** By the definition of  $T_o$ , Axiom 6.2, Lemma 10.1.4, and Proposition 10.1.1.  $\square$

Both versions of the Quasi-Equality Rules (analog of the Equality Rules (5201)) follow from Lemma 10.1.1 and Rule 1. By virtue of Lemmas 10.1.3 and the Quasi-Equality Rules, Rule 1 is valid if the hypothesis  $\mathbf{A}_\alpha \simeq \mathbf{B}_\alpha$  is replaced by  $\mathbf{B}_\alpha \simeq \mathbf{A}_\alpha$ ,  $\mathbf{A}_\alpha = \mathbf{B}_\alpha$ , or  $\mathbf{B}_\alpha = \mathbf{A}_\alpha$ ,

**Proposition 10.1.6**

1.  $\vdash^{\text{ef}} \mathbf{A}_o \downarrow$  where  $\mathbf{A}_o$  is evaluation-free.
2.  $\vdash \mathbf{A}_o \downarrow$ .

**Proof** By Axioms 6.1–3 and 6.5–8 for both parts.  $\square$

**Lemma 10.1.7** Let  $\mathbf{A}_\alpha$  and  $\mathbf{B}_\beta$  be evaluation-free. Either

$$\vdash^{\text{ef}} [\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner ] \uparrow$$

or

$$\vdash^{\text{ef}} \text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner = \ulcorner \mathbf{C}_\beta \urcorner$$

for some (evaluation-free) wff  $\mathbf{C}_\beta$ .

**Proof** Follows from Axiom 6.11, Axiom 10, Lemma 10.1.3, Specifications 7–9, the Tautology Theorem, and Rule 1.  $\square$

When  $\mathbf{A}_\alpha$  and  $\mathbf{B}_\beta$  are evaluation-free, let  $\mathbf{S}_{\mathbf{A}_\alpha}^{\mathbf{x}_\alpha} \mathbf{B}_\beta$  be the wff  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner$  denotes if  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner$  is defined and be undefined otherwise.

**Theorem 10.1.8 (Beta-Reduction Theorem: Analog of 5207)**

1.  $\vdash^{\text{ef}} \mathbf{A}_\alpha \downarrow \supset [\lambda \mathbf{x}_\alpha \mathbf{B}_\beta] \mathbf{A}_\alpha \simeq \mathbf{S}_{\mathbf{A}_\alpha}^{\mathbf{x}_\alpha} \mathbf{B}_\beta$ , provided  $\mathbf{S}_{\mathbf{A}_\alpha}^{\mathbf{x}_\alpha} \mathbf{B}_\beta$  is defined, where  $\mathbf{A}_\alpha$  and  $\mathbf{B}_\beta$  are evaluation-free.
2.  $\vdash [\mathbf{A}_\alpha \downarrow \wedge \text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_\beta \urcorner = \ulcorner \mathbf{C}_\beta \urcorner] \supset [\lambda \mathbf{x}_\alpha \mathbf{B}_\beta] \mathbf{A}_\alpha \simeq \mathbf{C}_\beta$ .

**Proof** Part 1 is by Axiom 4, Lemma 10.1.7, and the Tautology Theorem. Part 2 is immediately by Axiom 4.1.  $\square$

**Theorem 10.1.9 (Universal Instantiation: Analog of 5215)**

1. If  $\mathcal{H}^{\text{ef}} \vdash^{\text{ef}} \mathbf{A}_\alpha \downarrow$  and  $\mathcal{H}^{\text{ef}} \vdash^{\text{ef}} \forall \mathbf{x}_\alpha \mathbf{B}_o$ , then  $\mathcal{H}^{\text{ef}} \vdash^{\text{ef}} \mathbf{S}_{\mathbf{A}_\alpha}^{\mathbf{x}_\alpha} \mathbf{B}_o$ , provided  $\mathbf{S}_{\mathbf{A}_\alpha}^{\mathbf{x}_\alpha} \mathbf{B}_o$  is defined, where  $\mathbf{A}_\alpha$  and  $\mathbf{B}_\beta$  are evaluation-free.
2. If  $\mathcal{H} \vdash \mathbf{A}_\alpha \downarrow$ ,  $\mathcal{H} \vdash \text{sub}_{\epsilon\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \ulcorner \mathbf{x}_\alpha \urcorner \ulcorner \mathbf{B}_o \urcorner = \ulcorner \mathbf{C}_o \urcorner$ , and  $\mathcal{H} \vdash \forall \mathbf{x}_\alpha \mathbf{B}_o$ , then  $\mathcal{H} \vdash \mathbf{C}_o$ .
3. If  $\mathcal{H} \vdash [\lambda \mathbf{x}_\alpha \mathbf{B}_o] \mathbf{A}_\alpha = \mathbf{C}_o$  and  $\mathcal{H} \vdash \forall \mathbf{x}_\alpha \mathbf{B}_o$ , then  $\mathcal{H} \vdash \mathbf{C}_o$ .
4. If  $\mathcal{H} \vdash \forall \mathbf{x}_\alpha \mathbf{B}_o$ , then  $\mathcal{H} \vdash \mathbf{B}_o$ .

**Proof**

**Part 1**

$$\mathcal{H}^{\text{ef}} \vdash^{\text{ef}} \lambda \mathbf{x}_\alpha T_o = \lambda \mathbf{x}_\alpha \mathbf{B}_o. \quad (1)$$

$$\mathcal{H}^{\text{ef}} \vdash^{\text{ef}} [\lambda \mathbf{x}_\alpha T_o] \mathbf{A}_\alpha \simeq [\lambda \mathbf{x}_\alpha \mathbf{B}_o] \mathbf{A}_\alpha. \quad (2)$$

$$\mathcal{H}^{\text{ef}} \vdash^{\text{ef}} T_o \simeq \mathbf{S}_{\mathbf{A}_\alpha}^{\mathbf{x}_\alpha} \mathbf{B}_\beta. \quad (3)$$

$$\mathcal{H}^{\text{ef}} \vdash^{\text{ef}} \mathbf{S}_{\mathbf{A}_\alpha}^{\mathbf{x}_\alpha} \mathbf{B}_\beta. \quad (4)$$

(1) is by the definition of  $\forall$ ; (2) follows from (1) by the Quasi-Equality Rules; (3) follows from (2) by the first hypothesis, the Beta-Reduction Theorem (part 1), and Rule 1; and (4) follows from (3) and Corollary 10.1.5 by Rule 1.

**Part 2** Similar to Part 1.

**Part 3** Similar to Part 1.

**Part 4** Follows from Axiom 4.10, Lemma 10.1.4, and part 3 of this theorem.  $\square$

**Theorem 10.1.10 (Universal Generalization: Analog of 5220)**

1. If  $\mathcal{H}^{\text{ef}} \vdash^{\text{ef}} \mathbf{A}_o$ , then  $\mathcal{H}^{\text{ef}} \vdash^{\text{ef}} \forall \mathbf{x}_\alpha \mathbf{A}_o$  where  $\mathbf{A}_o$  is evaluation-free.
2. If  $\mathcal{H} \vdash \mathbf{A}_o$ , then  $\mathcal{H} \vdash \forall \mathbf{x}_\alpha \mathbf{A}_o$ .

**Proof**

**Part 1**

$$\mathcal{H}^{\text{ef}} \vdash^{\text{ef}} \mathbf{A}_o \quad (1)$$

$$\mathcal{H}^{\text{ef}} \vdash^{\text{ef}} T_o = \mathbf{A}_o \quad (2)$$

$$\mathcal{H}^{\text{ef}} \vdash^{\text{ef}} \lambda x_\alpha T_o = \lambda x_\alpha T_o \quad (3)$$

$$\mathcal{H}^{\text{ef}} \vdash^{\text{ef}} \forall \mathbf{x}_\alpha \mathbf{A}_o. \quad (4)$$

(1) is by hypothesis; (2) follows from (1) by the Tautology Theorem; (3) is by Axiom 6.5, Lemma 10.1.4, and Proposition 10.1.1; and (4) follows from (2) and (3) by Rule 1 and the definition of  $\forall$ .

**Part 2** Similar to Part 1.  $\square$

**Lemma 10.1.11 (Analog of 5209)** *If  $\vdash^{\text{ef}} \mathbf{A}_\alpha \downarrow$  and  $\vdash^{\text{ef}} \mathbf{B}_\beta \simeq \mathbf{C}_\beta$ , then  $\vdash^{\text{ef}} \mathbf{S}_{\mathbf{A}_\alpha}^{\mathbf{x}_\alpha}[\mathbf{B}_\beta \simeq \mathbf{C}_\beta]$ , provided  $\mathbf{S}_{\mathbf{A}_\alpha}^{\mathbf{x}_\alpha}[\mathbf{B}_\beta \simeq \mathbf{C}_\beta]$  is defined.*

**Proof** Similar to the proof of 5209 in [2]. It uses Proposition 10.1.1, the Beta-Reduction Theorem (part 1), and Rule 1.  $\square$

**Corollary 10.1.12** *If  $\vdash^{\text{ef}} \mathbf{A}_\alpha \downarrow$  and  $\vdash^{\text{ef}} \mathbf{B}_o = \mathbf{C}_o$ , then  $\vdash^{\text{ef}} \mathbf{S}_{\mathbf{A}_\alpha}^{\mathbf{x}_\alpha}[\mathbf{B}_o = \mathbf{C}_o]$ , provided  $\mathbf{S}_{\mathbf{A}_\alpha}^{\mathbf{x}_\alpha}[\mathbf{B}_o = \mathbf{C}_o]$  is defined.*

**Proof** By Lemma 10.1.3, Lemma 10.1.11, Proposition 10.1.6, and the Tautology Theorem.  $\square$

**Lemma 10.1.13 (Analog of 5205)**  $\vdash^{\text{ef}} f_{\alpha\beta} = \lambda \mathbf{y}_\beta [f_{\alpha\beta} \mathbf{y}_\beta]$ .

**Proof** Similar to the proof of 5205 in [2]. It uses Axiom 3, Axioms 6.1 and 6.5, Corollary 10.1.12, Lemmas 10.1.3 and 10.1.4, the Quasi-Equality Rules, the Beta-Reduction Theorem (part 1), and Rule 1.  $\square$

**Lemma 10.1.14 (Analog of 5206)**  $\vdash^{\text{ef}} \lambda \mathbf{x}_\beta \mathbf{A}_\alpha = \lambda \mathbf{z}_\beta \mathbf{S}_{\mathbf{z}_\beta}^{\mathbf{x}_\beta} \mathbf{A}_\alpha$ , provided  $\mathbf{z}_\beta$  is not free in  $\mathbf{A}_\alpha$  and  $\mathbf{S}_{\mathbf{z}_\beta}^{\mathbf{x}_\beta} \mathbf{A}_\alpha$  is defined.

**Proof** Similar to the proof of 5206 in [2]. It employs Axioms 6.1 and 6.5, Corollary 10.1.12, Lemma 10.1.13, the Beta-Reduction Theorem (part 1), and Rule 1.  $\square$

Analogs of  $\alpha$ -conversion,  $\beta$ -conversion, and  $\eta$ -conversion in [2] for evaluation-free proof are obtained directly from Lemma 10.1.14, the Beta-Reduction Theorem (part 1), Lemma 10.1.13 using Lemma 10.1.11 and Rule 1.

**Theorem 10.1.15 (Deduction Theorem: Analog of 5240)** *Let  $\mathbf{A}_o$  and  $\mathbf{H}_o$  be syntactically closed evaluation-free formulas. If  $\mathcal{H}^{\text{ef}} \cup \{\mathbf{H}_o\} \vdash^{\text{ef}} \mathbf{A}_o$ , then  $\mathcal{H}^{\text{ef}} \vdash^{\text{ef}} \mathbf{H}_o \supset \mathbf{A}_o$ .*

**Proof** Similar to the proof of 5240 in [2]. It uses Axioms 1–3 and 6, the Tautology Theorem, the Beta-Reduction Theorem (part 1), Universal Instantiation (part 1), Universal Generalization,  $\alpha$ -conversion, and Rule 1.  $\square$

## 10.2 Other Metatheorems

The metatheorems we prove in this subsection are not analogs of metatheorems of  $\mathcal{Q}_0$ ; they involve ordered pairs, quotation, and evaluation.

### Lemma 10.2.1 (Ordered Pairs)

1.  $\vdash^{\text{ef}} \forall x_\alpha \forall y_\beta [\text{pair}_{\langle \alpha \beta \rangle \beta \alpha} x_\alpha y_\beta] \downarrow$ .
2.  $\vdash^{\text{ef}} \forall x_\alpha \forall y_\beta [\text{fst}_{\langle \alpha \beta \rangle} [\text{pair}_{\langle \alpha \beta \rangle \beta \alpha} x_\alpha y_\beta] = x_\alpha]$ .
3.  $\vdash^{\text{ef}} \forall x_\alpha \forall y_\beta [\text{snd}_{\beta \langle \alpha \beta \rangle} [\text{pair}_{\langle \alpha \beta \rangle \beta \alpha} x_\alpha y_\beta] = y_\beta]$ .
4.  $\vdash^{\text{ef}} \forall z_{\langle \alpha \beta \rangle} [\text{pair}_{\langle \alpha \beta \rangle \beta \alpha} [\text{fst}_{\langle \alpha \beta \rangle} z_{\langle \alpha \beta \rangle}] [\text{snd}_{\beta \langle \alpha \beta \rangle} z_{\langle \alpha \beta \rangle}] = z_{\langle \alpha \beta \rangle}]$ .

**Proof** These four metatheorems of  $\mathcal{Q}_0^{\text{uqe}}$  can be straightforwardly proved using the definitions of  $\text{fst}_{\langle \alpha \beta \rangle}$  and  $\text{snd}_{\beta \langle \alpha \beta \rangle}$  and Axioms 8 and 9.  $\square$

**Theorem 10.2.2 (Injectiveness of Quotation)** *If  $\vdash^{\text{ef}} \ulcorner \mathbf{A}_\alpha \urcorner = \ulcorner \mathbf{B}_\alpha \urcorner$ , then  $\mathbf{A}_\alpha = \mathbf{B}_\alpha$ .*

**Proof** Assume  $\vdash^{\text{ef}} \ulcorner \mathbf{A}_\alpha \urcorner = \ulcorner \mathbf{B}_\alpha \urcorner$ . By Specification 1 and Rule 1, this implies  $\vdash^{\text{ef}} \mathcal{E}(\mathbf{A}_\alpha) = \mathcal{E}(\mathbf{B}_\alpha)$ . From this and Specifications 4.1–28, we can prove that  $\mathbf{A}_\alpha = \mathbf{B}_\alpha$  by induction on the size of  $\mathbf{A}_\alpha$ .  $\square$

**Theorem 10.2.3 (Disquotation Theorem)** *If  $D_\delta$  is evaluation-free, then  $\vdash \llbracket \ulcorner D_\delta \urcorner \rrbracket_\alpha \simeq D_\delta$ .*

**Proof** The proof is by induction on the size of  $\mathbf{D}_\delta$ .

**Case 1:**  $\mathbf{D}_\delta$  is  $\mathbf{x}_\alpha$ . Then  $\vdash \llbracket \ulcorner \mathbf{x}_\alpha \urcorner \rrbracket_\alpha = \mathbf{x}_\alpha$  by Axiom 11.1.

**Case 2:**  $\mathbf{D}_\delta$  is a primitive constant  $\mathbf{c}_\alpha$ . Then  $\vdash \llbracket \ulcorner \mathbf{c}_\alpha \urcorner \rrbracket_\alpha = \mathbf{c}_\alpha$  by Axiom 11.2.

**Case 3:**  $\mathbf{D}_\delta$  is  $\mathbf{A}_{\alpha\beta} \mathbf{B}_\beta$ . Assume (a)  $\mathbf{A}_{\alpha\beta} \mathbf{B}_\beta$  is evaluation-free. (a) implies (b)  $\mathbf{A}_{\alpha\beta}$  and  $\mathbf{B}_\beta$  are evaluation-free. Then we can derive the conclusion of the theorem as follows:

$$\vdash \llbracket \ulcorner \mathbf{A}_{\alpha\beta} \mathbf{B}_\beta \urcorner \rrbracket_\alpha \simeq \llbracket \ulcorner \mathbf{A}_{\alpha\beta} \mathbf{B}_\beta \urcorner \rrbracket_\alpha. \quad (1)$$

$$\vdash \llbracket \ulcorner \mathbf{A}_{\alpha\beta} \mathbf{B}_\beta \urcorner \rrbracket_\alpha \simeq \llbracket \mathcal{E}(\mathbf{A}_{\alpha\beta} \mathbf{B}_\beta) \rrbracket_\alpha. \quad (2)$$

$$\vdash \llbracket \ulcorner \mathbf{A}_{\alpha\beta} \mathbf{B}_\beta \urcorner \rrbracket_\alpha \simeq \llbracket \text{app}_{\epsilon\epsilon\epsilon} \mathcal{E}(\mathbf{A}_{\alpha\beta}) \mathcal{E}(\mathbf{B}_\beta) \rrbracket_\alpha. \quad (3)$$

$$\vdash \llbracket \ulcorner \mathbf{A}_{\alpha\beta} \mathbf{B}_\beta \urcorner \rrbracket_\alpha \simeq \llbracket \text{app}_{\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_{\alpha\beta} \urcorner \ulcorner \mathbf{B}_\beta \urcorner \rrbracket_\alpha. \quad (4)$$

$$\vdash \llbracket \text{app}_{\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_{\alpha\beta} \urcorner \ulcorner \mathbf{B}_\beta \urcorner \rrbracket_\alpha \simeq \llbracket \ulcorner \mathbf{A}_{\alpha\beta} \urcorner \rrbracket_{\alpha\beta} \llbracket \ulcorner \mathbf{B}_\beta \urcorner \rrbracket_\beta. \quad (5)$$

$$\vdash \llbracket \ulcorner \mathbf{A}_{\alpha\beta} \mathbf{B}_\beta \urcorner \rrbracket_\alpha \simeq \llbracket \ulcorner \mathbf{A}_{\alpha\beta} \urcorner \rrbracket_{\alpha\beta} \llbracket \ulcorner \mathbf{B}_\beta \urcorner \rrbracket_\beta. \quad (6)$$

$$\vdash \llbracket \ulcorner \mathbf{A}_{\alpha\beta} \mathbf{B}_\beta \urcorner \rrbracket_\alpha \simeq \mathbf{A}_{\alpha\beta} \mathbf{B}_\beta. \quad (7)$$

(1) is by Proposition 10.1.1; (2) and (4) follow from the (1) and (3), respectively, and Specification 1 by Rule 1; (3) follows from (2) by the definition of  $\mathcal{E}$ ; (5) is by Axiom 11.3; (6) follows from (4) and (5) by Rule 1; (7) follows from (b), the induction hypothesis, and (6) by Rule 1.

**Case 4:**  $\mathbf{D}_\delta$  is  $\lambda x_\beta \mathbf{A}_\alpha$ . Similar to Case 2. It is necessary to use the fact that  $\mathcal{E}(\mathbf{A}_\alpha)$  is semantically closed.

**Case 5:**  $\mathbf{D}_\delta$  is if  $\mathbf{A}_o \mathbf{B}_\alpha \mathbf{C}_\alpha$ . Similar to Case 2.

**Case 6:**  $\mathbf{D}_\delta$  is  $\ulcorner \mathbf{A}_\alpha \urcorner$ . Then we can derive the conclusion of the theorem as follows:

$$\vdash \llbracket \ulcorner \mathbf{A}_\alpha \urcorner \urcorner \rrbracket_\epsilon \simeq \llbracket \ulcorner \mathbf{A}_\alpha \urcorner \urcorner \rrbracket_\epsilon. \quad (1)$$

$$\vdash \llbracket \ulcorner \mathbf{A}_\alpha \urcorner \urcorner \rrbracket_\epsilon \simeq \llbracket \mathcal{E}(\ulcorner \mathbf{A}_\alpha \urcorner) \rrbracket_\epsilon. \quad (2)$$

$$\vdash \llbracket \ulcorner \mathbf{A}_\alpha \urcorner \urcorner \rrbracket_\epsilon \simeq \llbracket \text{quot}_{\epsilon\epsilon} \mathcal{E}(\mathbf{A}_\alpha) \rrbracket_\epsilon. \quad (3)$$

$$\vdash \llbracket \ulcorner \mathbf{A}_\alpha \urcorner \urcorner \rrbracket_\epsilon \simeq \llbracket \text{quot}_{\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \rrbracket_\epsilon. \quad (4)$$

$$\vdash \llbracket \text{quot}_{\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \rrbracket_\epsilon \simeq \text{if} \llbracket \text{quot}_{\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \rrbracket_\epsilon \downarrow \ulcorner \mathbf{A}_\alpha \urcorner \perp_\epsilon. \quad (5)$$

$$\vdash \llbracket \text{quot}_{\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \rrbracket_\epsilon \simeq \text{if} \llbracket \ulcorner \mathbf{A}_\alpha \urcorner \urcorner \rrbracket_\epsilon \downarrow \ulcorner \mathbf{A}_\alpha \urcorner \perp_\epsilon. \quad (6)$$

$$\vdash \llbracket \ulcorner \mathbf{A}_\alpha \urcorner \urcorner \rrbracket_\epsilon \downarrow. \quad (7)$$

$$\vdash \llbracket \text{quot}_{\epsilon\epsilon} \ulcorner \mathbf{A}_\alpha \urcorner \rrbracket_\epsilon \simeq \ulcorner \mathbf{A}_\alpha \urcorner. \quad (8)$$

$$\vdash \llbracket \ulcorner \mathbf{A}_\alpha \urcorner \urcorner \rrbracket_\epsilon \simeq \ulcorner \mathbf{A}_\alpha \urcorner. \quad (9)$$

(1) is by Proposition 10.1.1; (2) and (4) follow from the (1) and (3), respectively, and Specification 1 by Rule 1; (3) follows from (2) by the definition of  $\mathcal{E}$ ; (5) is by Axiom 11.6; (6) follows from (4) and (5) by Rule 1; (7) is by Axiom 6.8; (8) follows from (6) and (7) by Axiom 10.1 and Rule 1; and (9) follows from (4) and (8) by Rule 1.

**Case 7:**  $\mathbf{D}_\delta$  is  $\llbracket \mathbf{A}_\epsilon \rrbracket_\alpha$ . The theorem holds trivially in this case since  $\mathbf{D}_\delta$  is not evaluation-free.

□

## 11 Completeness

$\mathcal{P}^{\text{uqe}}$  is *complete* for  $\mathcal{Q}_0^{\text{uqe}}$  if  $\mathcal{H} \models_n \mathbf{A}_o$  implies  $\mathcal{H} \vdash \mathbf{A}_o$  whenever  $\mathbf{A}_o$  is a formula and  $\mathcal{H}$  is a set of sentences of  $\mathcal{Q}_0^{\text{uqe}}$ . However,  $\mathcal{P}^{\text{uqe}}$  is actually not complete for  $\mathcal{Q}_0^{\text{uqe}}$ . For instance, let  $A_o$  be the sentence

$$[\lambda x_\epsilon \lambda y_\epsilon \llbracket \text{app}_{\epsilon\epsilon} x_\epsilon y_\epsilon \rrbracket_\alpha] \ulcorner \sim_{oo} \urcorner T_o \urcorner = F_o.$$

Then, as observed in subsection 7.8,  $\models_n A_o$  holds but  $\vdash A_o$  does not hold.

$\mathcal{P}^{\text{uqe}}$  is *evaluation-free complete* for  $\mathcal{Q}_0^{\text{uqe}}$  if  $\mathcal{H} \models_n^{\text{ef}} \mathbf{A}_o$  implies  $\mathcal{H} \vdash^{\text{ef}} \mathbf{A}_o$  whenever  $\mathbf{A}_o$  is an evaluation-free formula and  $\mathcal{H}$  is a set of syntactically closed evaluation-free formulas of  $\mathcal{Q}_0^{\text{uqe}}$ . We will prove that  $\mathcal{P}^{\text{uqe}}$  is evaluation-free complete. Our proof will closely follow the proof of Theorem 22 (Henkin's Completeness Theorem for  $\mathcal{Q}_0^{\text{u}}$ ) in [23] which itself is based on the proof of 5502 (Henkin's Completeness and Soundness Theorem) in [2].

## 11.1 Extension Lemma

For any set  $S$ , let  $\text{card}(S)$  be the cardinality of  $S$ . Let  $\mathcal{L}(\mathcal{Q}_0^{\text{uqe}})$  be the set of wffs of  $\mathcal{Q}_0^{\text{uqe}}$ , let  $\kappa = \text{card}(\mathcal{L}(\mathcal{Q}_0^{\text{uqe}}))$ , let  $\mathcal{C}_\alpha$  be a well-ordered set of cardinality  $\kappa$  of new primitive constants of type  $\alpha$  for each  $\alpha \in \mathcal{T}$ , and let  $\mathcal{C} = \bigcup_{\alpha \in \mathcal{T}} \mathcal{C}_\alpha$ .

Define  $\overline{\mathcal{Q}_0^{\text{uqe}}}$  to be the logic that extends  $\mathcal{Q}_0^{\text{uqe}}$  as follows. The syntax of  $\overline{\mathcal{Q}_0^{\text{uqe}}}$  is obtained from the syntax of  $\mathcal{Q}_0^{\text{uqe}}$  by adding the members of  $\mathcal{C}$  to the primitive constants of  $\mathcal{Q}_0^{\text{uqe}}$  without extending the set of quotations of  $\mathcal{Q}_0^{\text{uqe}}$ . That is,  $\ulcorner \mathbf{c}_\alpha \urcorner$  is not a wff of  $\overline{\mathcal{Q}_0^{\text{uqe}}}$  for all  $\mathbf{c}_\alpha \in \mathcal{C}$ , and  $\mathcal{E}$  is still only defined on the wffs of  $\mathcal{Q}_0^{\text{uqe}}$ . Let  $\mathcal{L}(\overline{\mathcal{Q}_0^{\text{uqe}}})$  be the set of wffs of  $\overline{\mathcal{Q}_0^{\text{uqe}}}$ . Obviously,  $\text{card}(\mathcal{L}(\overline{\mathcal{Q}_0^{\text{uqe}}})) = \kappa$ . The semantics of  $\overline{\mathcal{Q}_0^{\text{uqe}}}$  is the same as the semantics of  $\mathcal{Q}_0^{\text{uqe}}$  except that a general or evaluation-free model for  $\overline{\mathcal{Q}_0^{\text{uqe}}}$  is a general or evaluation-free model  $\langle \{\mathcal{D}_\alpha \mid \alpha \in \mathcal{T}\}, \mathcal{J} \rangle$  for  $\mathcal{Q}_0^{\text{uqe}}$  where the domain of  $\mathcal{J}$  has been extended to include  $\mathcal{C}$ . Let  $\overline{\mathcal{P}^{\text{uqe}}}$  be the proof system that is obtained from  $\mathcal{P}^{\text{uqe}}$  by replacing the phrase ‘‘primitive constant’’ with the phrase ‘‘primitive constant not in  $\mathcal{C}$ ’’ in each formula schema in Specifications 1–9 and Axioms 1–12 except Axiom 6.2. Since  $\mathcal{L}(\overline{\mathcal{Q}_0^{\text{uqe}}})$  is a proper superset of  $\mathcal{L}(\mathcal{Q}_0^{\text{uqe}})$ , the axioms of  $\overline{\mathcal{P}^{\text{uqe}}}$  are a proper superset of the axioms of  $\mathcal{P}^{\text{uqe}}$ .  $\overline{\mathcal{P}^{\text{uqe}}}$  has the same rules of inference as  $\mathcal{P}^{\text{uqe}}$ . Let  $\mathcal{H} \vdash^{\text{ef}} \mathbf{A}_o$  mean there is an evaluation-free proof of  $\mathbf{A}_o$  from  $\mathcal{H}$  in  $\overline{\mathcal{P}^{\text{uqe}}}$ . Assume  $\overline{\mathcal{Q}_0^{\text{uqe}}}$  inherits all the other definitions of  $\mathcal{Q}_0^{\text{uqe}}$ .

An *xwff* of  $\overline{\mathcal{Q}_0^{\text{uqe}}}$  is a syntactically closed evaluation-free wff of  $\overline{\mathcal{Q}_0^{\text{uqe}}}$ . An *xwff* $_\alpha$  is an xwff of type  $\alpha$ . Let  $\mathcal{H}$  be a set of xwffs $_o$  of  $\overline{\mathcal{Q}_0^{\text{uqe}}}$ .  $\mathcal{H}$  is *evaluation-free complete* in  $\overline{\mathcal{P}^{\text{uqe}}}$  if, for every xwff $_o$   $\mathbf{A}_o$  of  $\overline{\mathcal{Q}_0^{\text{uqe}}}$ , either  $\mathcal{H} \vdash^{\text{ef}} \mathbf{A}_o$  or  $\mathcal{H} \vdash^{\text{ef}} \sim \mathbf{A}_o$ .  $\mathcal{H}$  is *evaluation-free extensionally complete* in  $\overline{\mathcal{P}^{\text{uqe}}}$  if, for every xwff $_o$  of the form  $\mathbf{A}_{\alpha\beta} = \mathbf{B}_{\alpha\beta}$  of  $\overline{\mathcal{Q}_0^{\text{uqe}}}$ , there is an xwff  $\mathbf{C}_\beta$  such that:

1.  $\mathcal{H} \vdash^{\text{ef}} \mathbf{C}_\beta \downarrow$ .
2.  $\mathcal{H} \vdash^{\text{ef}} [\mathbf{A}_{\alpha\beta} \downarrow \wedge \mathbf{B}_{\alpha\beta} \downarrow \wedge [\mathbf{A}_{\alpha\beta} \mathbf{C}_\beta \simeq \mathbf{B}_{\alpha\beta} \mathbf{C}_\beta]] \supset [\mathbf{A}_{\alpha\beta} = \mathbf{B}_{\alpha\beta}]$ .

**Lemma 11.1.1 (Extension Lemma)** *Let  $\mathcal{G}$  be a set of xwffs $_o$  of  $\mathcal{Q}_0^{\text{uqe}}$  consistent in  $\mathcal{P}^{\text{uqe}}$ . Then there is a set  $\mathcal{H}$  of xwffs $_o$  of  $\overline{\mathcal{Q}_0^{\text{uqe}}}$  such that:*

1.  $\mathcal{G} \subseteq \mathcal{H}$ .
2.  $\mathcal{H}$  is consistent in  $\overline{\mathcal{P}^{\text{uqe}}}$ .
3.  $\mathcal{H}$  is evaluation-free complete in  $\overline{\mathcal{P}^{\text{uqe}}}$ .
4.  $\mathcal{H}$  is evaluation-free extensionally complete in  $\overline{\mathcal{P}^{\text{uqe}}}$ .

**Proof** The proof is very close to the proof of 5500 in [2]. By transfinite induction, a set  $\mathcal{G}_\tau$  of xwffs $_\alpha$  is defined for each ordinal  $\tau \leq \kappa$ . The main difference between our proof and the proof of 5500 is that, in case (c) of the definition of  $\mathcal{G}_{\tau+1}$ ,

$$\mathcal{G}_{\tau+1} = \mathcal{G}_\tau \cup \{ \sim [\mathbf{A}_{\alpha\beta} \downarrow \wedge \mathbf{B}_{\alpha\beta} \downarrow \wedge [\mathbf{A}_{\alpha\beta} \mathbf{c}_\beta \simeq \mathbf{B}_{\alpha\beta} \mathbf{c}_\beta]] \}$$

where  $\mathbf{c}_\beta$  is the first constant in  $\mathcal{C}_\beta$  that does not occur in  $\mathcal{G}_\tau$  or  $\mathbf{A}_{\alpha\beta} = \mathbf{B}_{\alpha\beta}$ . (Notice that  $\overline{\vdash}^{\text{ef}} \mathbf{c}_\beta \downarrow$  by Axiom 6.2.)

To prove that  $\mathcal{G}_{\tau+1}$  is consistent in  $\overline{\mathcal{P}}^{\text{uqe}}$  assuming  $\mathcal{G}_\tau$  is consistent in  $\overline{\mathcal{P}}^{\text{uqe}}$  when  $\mathcal{G}_{\tau+1}$  is obtained by case (c), it is necessary to show that, if

$$\mathcal{G}_\tau \overline{\vdash}^{\text{ef}} \mathbf{A}_{\alpha\beta} \downarrow \wedge \mathbf{B}_{\alpha\beta} \downarrow \wedge [\mathbf{A}_{\alpha\beta} \mathbf{c}_\beta \simeq \mathbf{B}_{\alpha\beta} \mathbf{c}_\beta],$$

then  $\mathcal{G}_\tau \overline{\vdash}^{\text{ef}} \mathbf{A}_{\alpha\beta} = \mathbf{B}_{\alpha\beta}$ . Assume the hypothesis of this statement. Let  $P$  be an evaluation-free proof of

$$\mathbf{A}_{\alpha\beta} \downarrow \wedge \mathbf{B}_{\alpha\beta} \downarrow \wedge [\mathbf{A}_{\alpha\beta} \mathbf{c}_\beta \simeq \mathbf{B}_{\alpha\beta} \mathbf{c}_\beta]$$

from a finite subset  $\mathcal{S}$  of  $\mathcal{G}_\tau$ , and let  $\mathbf{x}_\beta$  be a variable that does not occur in  $P$  or  $\mathcal{S}$ . Since  $\mathbf{c}_\beta$  does not occur in  $\mathcal{G}_\tau$ ,  $\mathbf{A}_{\alpha\beta}$ , or  $\mathbf{B}_{\alpha\beta}$  and  $\mathbf{c}_\beta \in \mathcal{C}$ , the result of substituting  $\mathbf{x}_\beta$  for each occurrence of  $\mathbf{c}_\beta$  in  $P$  is an evaluation-free proof of

$$\mathbf{A}_{\alpha\beta} \downarrow \wedge \mathbf{B}_{\alpha\beta} \downarrow \wedge [\mathbf{A}_{\alpha\beta} \mathbf{x}_\beta \simeq \mathbf{B}_{\alpha\beta} \mathbf{x}_\beta]$$

from  $\mathcal{S}$ . Therefore,

$$\mathcal{S} \overline{\vdash}^{\text{ef}} \mathbf{A}_{\alpha\beta} \downarrow \wedge \mathbf{B}_{\alpha\beta} \downarrow \wedge [\mathbf{A}_{\alpha\beta} \mathbf{x}_\beta \simeq \mathbf{B}_{\alpha\beta} \mathbf{x}_\beta].$$

This implies

$$\mathcal{S} \overline{\vdash}^{\text{ef}} \mathbf{A}_{\alpha\beta} \downarrow, \mathcal{S} \overline{\vdash}^{\text{ef}} \mathbf{B}_{\alpha\beta} \downarrow, \mathcal{S} \overline{\vdash}^{\text{ef}} \forall \mathbf{x}_\beta [\mathbf{A}_{\alpha\beta} \mathbf{x}_\beta \simeq \mathbf{B}_{\alpha\beta} \mathbf{x}_\beta]$$

by the Tautology Theorem and Universal Generalization. It follows from these that  $\mathcal{G}_\tau \overline{\vdash}^{\text{ef}} \mathbf{A}_{\alpha\beta} = \mathbf{B}_{\alpha\beta}$  by Axiom 3, the Tautology Theorem, and Rule 1.

The rest of the proof is essentially the same as the proof of 5500.  $\square$

## 11.2 Henkin's Theorem

A general or evaluation-free model  $\langle \{\mathcal{D}_\alpha \mid \alpha \in \mathcal{T}\}, \mathcal{J} \rangle$  for  $\mathcal{Q}_0^{\text{uqe}}$  is *frugal* if  $\text{card}(\mathcal{D}_\alpha) \leq \text{card}(\mathcal{L}(\mathcal{Q}_0^{\text{uqe}}))$  for all  $\alpha \in \mathcal{T}$ .

**Theorem 11.2.1 (Henkin's Theorem for  $\mathcal{P}^{\text{uqe}}$ )** *Every set of syntactically closed evaluation-free formulas of  $\mathcal{Q}_0^{\text{uqe}}$  consistent in  $\mathcal{P}^{\text{uqe}}$  has a frugal normal evaluation-free model.*

**Proof** The proof is very close to the proof of Theorem 21 in [23]. Let  $\mathcal{G}$  be a set of  $\text{xwffs}_\alpha$  of  $\mathcal{Q}_0^{\text{uqe}}$  consistent in  $\mathcal{P}^{\text{uqe}}$ , and let  $\mathcal{H}$  be a set of  $\text{xwffs}_\alpha$  of  $\overline{\mathcal{Q}}_0^{\text{uqe}}$  that satisfies the four statements of the Extension Lemma.

**Step 1** We define simultaneously, by recursion on  $\gamma \in \mathcal{T}$ , a frame  $\{\mathcal{D}_\alpha \mid \alpha \in \mathcal{T}\}$  and a partial function  $\mathcal{V}$  whose domain is the set of  $\text{xwffs}$  of  $\overline{\mathcal{Q}}_0^{\text{uqe}}$  so that the following conditions hold for all  $\gamma \in \mathcal{T}$ :

$$(1^\gamma) \mathcal{D}_\gamma = \{\mathcal{V}(\mathbf{A}_\gamma) \mid \mathbf{A}_\gamma \text{ is a } \text{xwff}_\gamma \text{ and } \mathcal{H} \overline{\vdash}^{\text{ef}} \mathbf{A}_\gamma \downarrow\}.$$



(2<sup>γ</sup>)  $\mathcal{V}(\mathbf{A}_\gamma)$  is defined iff  $\mathcal{H} \overline{\vdash}^{\text{ef}} \mathbf{A}_\gamma \downarrow$  for all xwffs  $\mathbf{A}_\gamma$ .

(3<sup>γ</sup>)  $\mathcal{V}(\mathbf{A}_\gamma) = \mathcal{V}(\mathbf{B}_\gamma)$  iff  $\mathcal{H} \overline{\vdash}^{\text{ef}} \mathbf{A}_\gamma = \mathbf{B}_\gamma$  for all xwffs  $\mathbf{A}_\gamma$  and  $\mathbf{B}_\gamma$ .

Let  $\mathcal{V}(x) \simeq \mathcal{V}(y)$  mean either  $\mathcal{V}(x)$  and  $\mathcal{V}(y)$  are both defined and equal or  $\mathcal{V}(x)$  and  $\mathcal{V}(y)$  are both undefined.

**Step 1.1** We define  $\mathcal{D}_i$  and  $\mathcal{V}$  on xwffs<sub>*i*</sub>. For each xwff  $\mathbf{A}_i$ , if  $\mathcal{H} \overline{\vdash}^{\text{ef}} \mathbf{A}_i \downarrow$ , let

$$\mathcal{V}(\mathbf{A}_i) = \{\mathbf{B}_i \mid \mathbf{B}_i \text{ is a xwff}_i \text{ and } \mathcal{H} \overline{\vdash}^{\text{ef}} \mathbf{A}_i = \mathbf{B}_i\},$$

and otherwise let  $\mathcal{V}(\mathbf{A}_i)$  be undefined. Also, let

$$\mathcal{D}_i = \{\mathcal{V}(\mathbf{A}_i) \mid \mathbf{A}_i \text{ is a xwff}_i \text{ and } \mathcal{H} \overline{\vdash}^{\text{ef}} \mathbf{A}_i \downarrow\}.$$

(1<sup>*i*</sup>), (2<sup>*i*</sup>), and (3<sup>*i*</sup>) are clearly satisfied.

**Step 1.2** We define  $\mathcal{D}_o$  and  $\mathcal{V}$  on xwffs<sub>*o*</sub>. For each xwff  $\mathbf{A}_o$ , if  $\mathcal{H} \overline{\vdash}^{\text{ef}} \mathbf{A}_o$ , let  $\mathcal{V}(\mathbf{A}_o) = \text{T}$ , and otherwise let  $\mathcal{V}(\mathbf{A}_o) = \text{F}$ . Also, let  $\mathcal{D}_o = \{\text{T}, \text{F}\}$ . By the consistency and evaluation-free completeness of  $\mathcal{H}$ , exactly one of  $\mathcal{H} \overline{\vdash}^{\text{ef}} \mathbf{A}_o$  and  $\mathcal{H} \overline{\vdash}^{\text{ef}} \sim \mathbf{A}_o$  holds. By Proposition 10.1.6,  $\mathcal{H} \overline{\vdash}^{\text{ef}} \mathbf{A}_o \downarrow$  for all wffs  $\mathbf{A}_o$ . Hence (1<sup>*o*</sup>), (2<sup>*o*</sup>), and (3<sup>*o*</sup>) are satisfied.

**Step 1.3** We define  $\mathcal{D}_\epsilon$  and  $\mathcal{V}$  on xwffs<sub>*ε*</sub>. Let

$$\mathcal{D}_\epsilon = \{\mathcal{E}(\mathbf{A}_\alpha) \mid \mathbf{A}_\alpha \text{ is a wff of } \mathcal{Q}_0^{\text{uqe}}\}.$$

Choose a mapping  $f$  from  $\{\mathbf{A}_\epsilon \mid \mathbf{A}_\epsilon \text{ is an xwff}_\epsilon \text{ and } \mathcal{H} \overline{\vdash}^{\text{ef}} \mathbf{A}_\epsilon \downarrow\}$  to  $\mathcal{D}_\epsilon$  such that:

1.  $f(\mathbf{A}_\epsilon) = f(\mathbf{B}_\epsilon)$  iff  $\mathcal{H} \overline{\vdash}^{\text{ef}} \mathbf{A}_\epsilon = \mathbf{B}_\epsilon$ .
2. If  $\mathcal{H} \overline{\vdash}^{\text{ef}} \mathbf{A}_\epsilon = \mathcal{E}(\mathbf{C}_\gamma)$ , then  $f(\mathbf{A}_\epsilon) = \mathcal{E}(\mathbf{C}_\gamma)$ .
3. If  $\mathcal{H} \overline{\vdash}^{\text{ef}} \text{wff}_{\alpha\epsilon}^\alpha \mathbf{A}_\epsilon$ , then  $f(\mathbf{A}_\epsilon) = \mathcal{E}(\mathbf{C}_\alpha)$  for some wff  $\mathbf{C}_\alpha$ .

It is possible to choose such a mapping by Lemma 10.2.2, Specification 6.13, and the fact that  $\text{card}(\mathcal{L}(\mathcal{Q}_0^{\text{uqe}})) = \text{card}(\mathcal{L}(\mathcal{Q}_0^{\text{uqe}}))$ . For each xwff  $\mathbf{A}_\epsilon$ , if  $\mathcal{H} \overline{\vdash}^{\text{ef}} \mathbf{A}_\epsilon \downarrow$ , let  $\mathcal{V}(\mathbf{A}_\epsilon) = f(\mathbf{A}_\epsilon)$ , and otherwise let  $\mathcal{V}(\mathbf{A}_\epsilon)$  be undefined. (2<sup>*ε*</sup>) and (3<sup>*ε*</sup>) are clearly satisfied; (1<sup>*ε*</sup>) is satisfied since, for all wffs  $\mathbf{A}_\alpha$  of  $\mathcal{Q}_0^{\text{uqe}}$ ,  $\mathcal{E}(\mathbf{A}_\alpha)$  is an xwff<sub>*ε*</sub> by the definition of  $\mathcal{E}$  and Specification 7 and  $\mathcal{H} \overline{\vdash}^{\text{ef}} \mathcal{E}(\mathbf{A}_\alpha) \downarrow$  by Axiom 6.7 and Specification 1.

**Step 1.4** We define  $\mathcal{D}_{\alpha\beta}$  and  $\mathcal{V}$  on xwffs<sub>*αβ*</sub> for all  $\alpha, \beta \in \mathcal{T}$ . Now suppose that  $\mathcal{D}_\alpha$  and  $\mathcal{D}_\beta$  are defined and that the conditions hold for  $\alpha$  and  $\beta$ . For each xwff  $\mathbf{A}_{\alpha\beta}$ , if  $\mathcal{H} \overline{\vdash}^{\text{ef}} \mathbf{A}_{\alpha\beta} \downarrow$ , let  $\mathcal{V}(\mathbf{A}_{\alpha\beta})$  be the (partial or total) function from  $\mathcal{D}_\beta$  to  $\mathcal{D}_\alpha$  whose value, for any argument  $\mathcal{V}(\mathbf{B}_\beta) \in \mathcal{D}_\beta$ , is  $\mathcal{V}(\mathbf{A}_{\alpha\beta}\mathbf{B}_\beta)$  if  $\mathcal{V}(\mathbf{A}_{\alpha\beta}\mathbf{B}_\beta)$  is defined and is undefined if  $\mathcal{V}(\mathbf{A}_{\alpha\beta}\mathbf{B}_\beta)$  is undefined, and otherwise let  $\mathcal{V}(\mathbf{A}_{\alpha\beta})$  be undefined. We must show that this definition is independent of the particular xwff  $\mathbf{B}_\beta$  used to represent the argument. So suppose  $\mathcal{V}(\mathbf{B}_\beta) = \mathcal{V}(\mathbf{C}_\beta)$ ; then

$\mathcal{H} \overline{\vdash}^{\text{ef}} \mathbf{B}_\beta = \mathbf{C}_\beta$  by  $(3^\beta)$ , so  $\mathcal{H} \overline{\vdash}^{\text{ef}} \mathbf{A}_{\alpha\beta} \mathbf{B}_\beta \simeq \mathbf{A}_{\alpha\beta} \mathbf{C}_\beta$  by Lemma 10.1.3 and the Quasi-Equality Rules, and so  $\mathcal{V}(\mathbf{A}_{\alpha\beta} \mathbf{B}_\beta) \simeq \mathcal{V}(\mathbf{A}_{\alpha\beta} \mathbf{C}_\beta)$  by  $(2^\alpha)$  and  $(3^\alpha)$ . Finally, let

$$\mathcal{D}_{\alpha\beta} = \{\mathcal{V}(\mathbf{A}_{\alpha\beta}) \mid \mathbf{A}_{\alpha\beta} \text{ is a xwff}_{\alpha\beta} \text{ and } \mathcal{H} \overline{\vdash}^{\text{ef}} \mathbf{A}_{\alpha\beta} \downarrow\}.$$

$(1^{\alpha\beta})$  and  $(2^{\alpha\beta})$  are clearly satisfied; we must show that  $(3^{\alpha\beta})$  is satisfied. Suppose  $\mathcal{V}(\mathbf{A}_{\alpha\beta}) = \mathcal{V}(\mathbf{B}_{\alpha\beta})$ . Then  $\mathcal{H} \overline{\vdash}^{\text{ef}} \mathbf{A}_{\alpha\beta} \downarrow$  and  $\mathcal{H} \overline{\vdash}^{\text{ef}} \mathbf{B}_{\alpha\beta} \downarrow$ . Since  $\mathcal{H}$  is evaluation-free extensionally complete, there is a  $\mathbf{C}_\beta$  such that  $\mathcal{H} \overline{\vdash}^{\text{ef}} \mathbf{C}_\beta \downarrow$  and

$$\mathcal{H} \overline{\vdash}^{\text{ef}} [\mathbf{A}_{\alpha\beta} \downarrow \wedge \mathbf{B}_{\alpha\beta} \downarrow \wedge [\mathbf{A}_{\alpha\beta} \mathbf{C}_\beta \simeq \mathbf{B}_{\alpha\beta} \mathbf{C}_\beta]] \supset [\mathbf{A}_{\alpha\beta} = \mathbf{B}_{\alpha\beta}].$$

Then  $\mathcal{V}(\mathbf{A}_{\alpha\beta} \mathbf{C}_\beta) \simeq \mathcal{V}(\mathbf{A}_{\alpha\beta})(\mathcal{V}(\mathbf{C}_\beta)) \simeq \mathcal{V}(\mathbf{B}_{\alpha\beta})(\mathcal{V}(\mathbf{C}_\beta)) \simeq \mathcal{V}(\mathbf{B}_{\alpha\beta} \mathbf{C}_\beta)$ , so  $\mathcal{H} \overline{\vdash}^{\text{ef}} \mathbf{A}_{\alpha\beta} \mathbf{C}_\beta \simeq \mathbf{B}_{\alpha\beta} \mathbf{C}_\beta$  by  $(2^\alpha)$  and  $(3^\alpha)$ , and so  $\mathcal{H} \overline{\vdash}^{\text{ef}} \mathbf{A}_{\alpha\beta} = \mathbf{B}_{\alpha\beta}$ . Now suppose  $\mathcal{H} \overline{\vdash}^{\text{ef}} \mathbf{A}_{\alpha\beta} = \mathbf{B}_{\alpha\beta}$ . Then, for all xwffs  $\mathbf{C}_\beta \in \mathcal{D}_\beta$ ,  $\mathcal{H} \overline{\vdash}^{\text{ef}} \mathbf{A}_{\alpha\beta} \mathbf{C}_\beta \simeq \mathbf{B}_{\alpha\beta} \mathbf{C}_\beta$  by Lemma 10.1.3 and the Quasi-Equality Rules, and so  $\mathcal{V}(\mathbf{A}_{\alpha\beta})(\mathcal{V}(\mathbf{C}_\beta)) \simeq \mathcal{V}(\mathbf{A}_{\alpha\beta} \mathbf{C}_\beta) \simeq \mathcal{V}(\mathbf{B}_{\alpha\beta} \mathbf{C}_\beta) \simeq \mathcal{V}(\mathbf{B}_{\alpha\beta})(\mathcal{V}(\mathbf{C}_\beta))$ . Hence  $\mathcal{V}(\mathbf{A}_{\alpha\beta}) = \mathcal{V}(\mathbf{B}_{\alpha\beta})$ .

**Step 1.5** We define  $\mathcal{D}_{\langle\alpha\beta\rangle}$  and  $\mathcal{V}$  on xwffs $_{\langle\alpha\beta\rangle}$  for all  $\alpha, \beta \in \mathcal{T}$ . Now suppose that  $\mathcal{D}_\alpha$  and  $\mathcal{D}_\beta$  are defined and that the conditions hold for  $\alpha$  and  $\beta$ . For each xwff  $\mathbf{A}_{\langle\alpha\beta\rangle}$ , if  $\mathcal{H} \overline{\vdash}^{\text{ef}} \mathbf{A}_{\langle\alpha\beta\rangle} \downarrow$ , let

$$\mathcal{V}(\mathbf{A}_{\langle\alpha\beta\rangle}) = \langle \mathcal{V}(\text{fst}_{\alpha\langle\alpha\beta\rangle} \mathbf{A}_{\langle\alpha\beta\rangle}), \mathcal{V}(\text{snd}_{\beta\langle\alpha\beta\rangle} \mathbf{A}_{\langle\alpha\beta\rangle}) \rangle,$$

and otherwise let  $\mathcal{V}(\mathbf{A}_{\langle\alpha\beta\rangle})$  be undefined. Also, let

$$\mathcal{D}_{\langle\alpha\beta\rangle} = \{\mathcal{V}(\mathbf{A}_{\langle\alpha\beta\rangle}) \mid \mathbf{A}_{\langle\alpha\beta\rangle} \text{ is a xwff}_{\langle\alpha\beta\rangle} \text{ and } \mathcal{H} \overline{\vdash}^{\text{ef}} \mathbf{A}_{\langle\alpha\beta\rangle} \downarrow\}.$$

$(1^{\langle\alpha\beta\rangle})$  and  $(2^{\langle\alpha\beta\rangle})$  are clearly satisfied; we must show that  $(3^{\langle\alpha\beta\rangle})$  is satisfied.

$$\mathcal{V}(\mathbf{A}_{\langle\alpha\beta\rangle}) = \mathcal{V}(\mathbf{B}_{\langle\alpha\beta\rangle}) \tag{1}$$

$$\text{iff } \langle \langle \mathcal{V}(\text{fst}_{\alpha\langle\alpha\beta\rangle} \mathbf{A}_{\langle\alpha\beta\rangle}), \mathcal{V}(\text{snd}_{\beta\langle\alpha\beta\rangle} \mathbf{A}_{\langle\alpha\beta\rangle}) \rangle, \langle \mathcal{V}(\text{fst}_{\alpha\langle\alpha\beta\rangle} \mathbf{B}_{\langle\alpha\beta\rangle}), \mathcal{V}(\text{snd}_{\beta\langle\alpha\beta\rangle} \mathbf{B}_{\langle\alpha\beta\rangle}) \rangle \rangle \tag{2}$$

$$\text{iff } \mathcal{V}(\text{fst}_{\alpha\langle\alpha\beta\rangle} \mathbf{A}_{\langle\alpha\beta\rangle}) = \mathcal{V}(\text{fst}_{\alpha\langle\alpha\beta\rangle} \mathbf{B}_{\langle\alpha\beta\rangle}) \text{ and } \mathcal{V}(\text{snd}_{\beta\langle\alpha\beta\rangle} \mathbf{A}_{\langle\alpha\beta\rangle}) = \mathcal{V}(\text{snd}_{\beta\langle\alpha\beta\rangle} \mathbf{B}_{\langle\alpha\beta\rangle}) \tag{3}$$

$$\text{iff } \mathcal{H} \overline{\vdash}^{\text{ef}} \text{fst}_{\alpha\langle\alpha\beta\rangle} \mathbf{A}_{\langle\alpha\beta\rangle} = \text{fst}_{\alpha\langle\alpha\beta\rangle} \mathbf{B}_{\langle\alpha\beta\rangle} \text{ and } \mathcal{H} \overline{\vdash}^{\text{ef}} \text{snd}_{\beta\langle\alpha\beta\rangle} \mathbf{A}_{\langle\alpha\beta\rangle} = \text{snd}_{\beta\langle\alpha\beta\rangle} \mathbf{B}_{\langle\alpha\beta\rangle} \tag{4}$$

$$\text{iff } \mathcal{H} \overline{\vdash}^{\text{ef}} \text{pair}_{\langle\alpha\beta\rangle\beta\alpha} [\text{fst}_{\alpha\langle\alpha\beta\rangle} \mathbf{A}_{\langle\alpha\beta\rangle}] [\text{snd}_{\beta\langle\alpha\beta\rangle} \mathbf{A}_{\langle\alpha\beta\rangle}] = \text{pair}_{\langle\alpha\beta\rangle\beta\alpha} [\text{fst}_{\alpha\langle\alpha\beta\rangle} \mathbf{B}_{\langle\alpha\beta\rangle}] [\text{snd}_{\beta\langle\alpha\beta\rangle} \mathbf{B}_{\langle\alpha\beta\rangle}] \tag{5}$$

$$\text{iff } \mathcal{H} \overline{\vdash}^{\text{ef}} \mathbf{A}_{\langle\alpha\beta\rangle} = \mathbf{B}_{\langle\alpha\beta\rangle}. \tag{6}$$

(2) is by the definition of  $\mathcal{V}$  on xwffs $_{\langle\alpha\beta\rangle}$ ; (3) is by definition of ordered pairs; (4) is by  $(3^\alpha)$  and  $(3^\beta)$ ; (5) is by Axiom 9.1; and (6) is by Axioms 9.1 and 9.2. Hence  $(3^{\langle\alpha\beta\rangle})$  is satisfied.

**Step 2** We claim that  $\mathcal{M} = \langle \{\mathcal{D}_\alpha \mid \alpha \in \mathcal{T}\}, \mathcal{V} \rangle$  is an interpretation. For each primitive constant  $\mathbf{c}_\gamma$  of  $\overline{\mathcal{Q}}_0^{\text{uqe}}$ ,  $\mathbf{c}_\gamma$  is an  $\text{xwff}_\gamma$  and  $\mathcal{H} \vdash^{\text{ef}} \mathbf{c}_\gamma \downarrow$  by Axiom 6.2, and thus  $\mathcal{V}$  maps each primitive constant of  $\overline{\mathcal{Q}}_0^{\text{uqe}}$  of type  $\gamma$  into  $\mathcal{D}_\gamma$  by  $(1^\gamma)$  and  $(2^\gamma)$ .

**Step 2.1** We must show  $\mathcal{V}(\mathbf{Q}_{o\alpha\alpha}) = \mathcal{J}(\mathbf{Q}_{o\alpha\alpha})$ , i.e., that  $\mathcal{V}(\mathbf{Q}_{o\alpha\alpha})$  is the identity relation on  $\mathcal{D}_\alpha$ . Let  $\mathcal{V}(\mathbf{A}_\alpha)$  and  $\mathcal{V}(\mathbf{B}_\alpha)$  be arbitrary members of  $\mathcal{D}_\alpha$ . Then  $\mathcal{V}(\mathbf{A}_\alpha) = \mathcal{V}(\mathbf{B}_\alpha)$  iff  $\mathcal{H} \vdash^{\text{ef}} \mathbf{A}_\alpha = \mathbf{B}_\alpha$  iff  $\mathcal{H} \vdash^{\text{ef}} \mathbf{Q}_{o\alpha\alpha} \mathbf{A}_\alpha \mathbf{B}_\alpha$  iff  $\top = \mathcal{V}(\mathbf{Q}_{o\alpha\alpha} \mathbf{A}_\alpha \mathbf{B}_\alpha) = \mathcal{V}(\mathbf{Q}_{o\alpha\alpha})(\mathcal{V}(\mathbf{A}_\alpha))(\mathcal{V}(\mathbf{B}_\alpha))$ . Thus  $\mathcal{V}(\mathbf{Q}_{o\alpha\alpha})$  is the identity relation on  $\mathcal{D}_\alpha$ .

**Step 2.2** We must show that  $\mathcal{V}(\iota_{\alpha(o\alpha)}) = \mathcal{J}(\iota_{\alpha(o\alpha)})$ , i.e., that, for  $\alpha \neq o$ ,  $\mathcal{V}(\iota_{\alpha(o\alpha)})$  is the unique member selector on  $\mathcal{D}_\alpha$ . For  $\alpha \neq o$ , let  $\mathcal{V}(\mathbf{A}_{o\alpha})$  be an arbitrary member of  $\mathcal{D}_{o\alpha}$  and  $\mathbf{x}_\alpha$  be a variable that does not occur in  $\mathbf{A}_{o\alpha}$ . Suppose  $\mathcal{V}(\mathbf{A}_{o\alpha}) = \mathcal{V}(\mathbf{Q}_{o\alpha\alpha} \mathbf{B}_\alpha)$ . We must show that  $\mathcal{V}(\iota_{\alpha(o\alpha)})(\mathcal{V}(\mathbf{A}_{o\alpha})) = \mathcal{V}(\mathbf{B}_\alpha)$ . The hypothesis implies  $\mathcal{H} \vdash^{\text{ef}} \mathbf{A}_{o\alpha} = \mathbf{Q}_{o\alpha\alpha} \mathbf{B}_\alpha$ , so  $\mathcal{H} \vdash^{\text{ef}} \exists_1 \mathbf{x}_\alpha [\mathbf{A}_{o\alpha} \mathbf{x}_\alpha]$  by the definition of  $\exists_1$ , and so  $\mathcal{H} \vdash^{\text{ef}} \mathbf{A}_{o\alpha} [\mathbf{I}\mathbf{x}_\alpha \mathbf{A}_{o\alpha}]$  by Axiom 8.1 and Axiom 8.2. Hence  $\mathcal{H} \vdash^{\text{ef}} \mathbf{Q}_{o\alpha\alpha} \mathbf{B}_\alpha [\mathbf{I}\mathbf{x}_\alpha \mathbf{A}_{o\alpha}]$  by Rule 1, and so  $\mathcal{V}(\mathbf{B}_\alpha) = \mathcal{V}(\mathbf{I}\mathbf{x}_\alpha \mathbf{A}_{o\alpha}) = \mathcal{V}(\iota_{\alpha(o\alpha)} \mathbf{A}_{o\alpha}) = \mathcal{V}(\iota_{\alpha(o\alpha)})(\mathcal{V}(\mathbf{A}_{o\alpha}))$ .

Now suppose that  $\mathcal{V}(\forall \mathbf{x}_\alpha [\mathbf{A}_{o\alpha} \neq \mathbf{Q}_{o\alpha\alpha} \mathbf{x}_\alpha]) = \top$ . We must show that  $\mathcal{V}(\iota_{\alpha(o\alpha)})(\mathcal{V}(\mathbf{A}_{o\alpha}))$  is undefined. The hypothesis implies  $\mathcal{H} \vdash^{\text{ef}} \forall \mathbf{x}_\alpha [\mathbf{A}_{o\alpha} \neq \mathbf{Q}_{o\alpha\alpha} \mathbf{x}_\alpha]$ , so  $\mathcal{H} \vdash^{\text{ef}} \sim [\exists_1 \mathbf{x}_\alpha [\mathbf{A}_{o\alpha} \mathbf{x}_\alpha]]$  by the definition of  $\exists_1$ , and so  $\mathcal{H} \vdash^{\text{ef}} [\mathbf{I}\mathbf{x}_\alpha [\mathbf{A}_{o\alpha} \mathbf{x}_\alpha]] \uparrow$  by Axiom 8.1. Hence  $\mathcal{V}(\mathbf{I}\mathbf{x}_\alpha [\mathbf{A}_{o\alpha} \mathbf{x}_\alpha]) \simeq \mathcal{V}(\iota_{\alpha(o\alpha)} \mathbf{A}_{o\alpha}) \simeq \mathcal{V}(\iota_{\alpha(o\alpha)})(\mathcal{V}(\mathbf{A}_{o\alpha}))$  is undefined.

**Step 2.4** We must show that  $\mathcal{V}(\text{pair}_{\langle\alpha\beta\rangle\beta\alpha}) = \mathcal{J}(\text{pair}_{\langle\alpha\beta\rangle\beta\alpha})$ . Let  $\mathcal{V}(\mathbf{A}_\alpha)$  be an arbitrary member of  $\mathcal{D}_\alpha$  and  $\mathcal{V}(\mathbf{B}_\beta)$  be an arbitrary member of  $\mathcal{D}_\beta$ . We must show that  $\mathcal{V}(\text{pair}_{\langle\alpha\beta\rangle\beta\alpha} \mathbf{A}_\alpha, \mathbf{B}_\beta) = \langle \mathcal{V}(\mathbf{A}_\alpha), \mathcal{V}(\mathbf{B}_\beta) \rangle$ .  $\mathcal{V}(\text{fst}_{\alpha\langle\alpha\beta\rangle} [\text{pair}_{\langle\alpha\beta\rangle\beta\alpha} \mathbf{A}_\alpha, \mathbf{B}_\beta]) = \mathcal{V}(\mathbf{A}_\alpha)$  iff  $\mathcal{H} \vdash^{\text{ef}} \text{fst}_{\alpha\langle\alpha\beta\rangle} [\text{pair}_{\langle\alpha\beta\rangle\beta\alpha} \mathbf{A}_\alpha, \mathbf{B}_\beta] = \mathbf{A}_\alpha$ , which holds by the definition of  $\text{fst}_{\alpha\langle\alpha\beta\rangle}$  and Axiom 9.1. Similarly,  $\mathcal{V}(\text{snd}_{\beta\langle\alpha\beta\rangle} [\text{pair}_{\langle\alpha\beta\rangle\beta\alpha} \mathbf{A}_\alpha, \mathbf{B}_\beta]) = \mathcal{V}(\mathbf{B}_\beta)$  holds by the definition of  $\text{snd}_{\beta\langle\alpha\beta\rangle}$  and Axiom 9.1. Hence

$$\begin{aligned} & \mathcal{V}(\text{pair}_{\langle\alpha\beta\rangle\beta\alpha} \mathbf{A}_\alpha, \mathbf{B}_\beta) \\ &= \langle \mathcal{V}(\text{fst}_{\alpha\langle\alpha\beta\rangle} [\text{pair}_{\langle\alpha\beta\rangle\beta\alpha} \mathbf{A}_\alpha, \mathbf{B}_\beta]), \mathcal{V}(\text{snd}_{\beta\langle\alpha\beta\rangle} [\text{pair}_{\langle\alpha\beta\rangle\beta\alpha} \mathbf{A}_\alpha, \mathbf{B}_\beta]) \rangle \\ &= \langle \mathcal{V}(\mathbf{A}_\alpha), \mathcal{V}(\mathbf{B}_\beta) \rangle. \end{aligned}$$

Thus  $\mathcal{M}$  is an interpretation.

**Step 3** We claim further that  $\mathcal{M}$  is an evaluation-free model for  $\overline{\mathcal{Q}}_0^{\text{uqe}}$ . For each assignment  $\varphi \in \text{assign}(\mathcal{M})$  and evaluation-free wff  $\mathbf{D}_\delta$ , let

$$\mathbf{D}_\delta^\varphi = S_{\mathbf{E}_{\delta_1}^1 \dots \mathbf{E}_{\delta_n}^n} \mathbf{D}_\delta = S_{\mathbf{E}_{\delta_1}^1} S_{\mathbf{E}_{\delta_2}^2} \dots S_{\mathbf{E}_{\delta_n}^n} \mathbf{D}_\delta$$

where  $\mathbf{x}_{\delta_1}^1 \dots \mathbf{x}_{\delta_n}^n$  are the free variables of  $\mathbf{D}_\delta$  and  $\mathbf{E}_{\delta_i}^i$  is the first xwff (in some fixed enumeration) of  $\mathcal{L}(\overline{\mathcal{Q}}_0^{\text{uqe}})$  such that  $\varphi(\mathbf{x}_{\delta_i}^i) = \mathcal{V}(\mathbf{E}_{\delta_i}^i)$  for all  $i$  with  $1 \leq i \leq n$ . Since each  $\mathbf{E}_{\delta_i}^i$  is syntactically closed,  $\mathbf{D}_\delta^\varphi$  is always defined.

Let  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta) \simeq \mathcal{V}(\mathbf{D}_\delta^\varphi)$ .  $\mathbf{D}_\delta^\varphi$  is clearly a xwff $_\delta$ , so  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta) \in \mathcal{D}_\delta$  if  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta)$  is defined. We will show that the six conditions of an evaluation-free model are satisfied as follows:

1. Let  $\mathbf{D}_\delta$  be a variable  $\mathbf{x}_\delta$ . Choose  $\mathbf{E}_\delta$  so that  $\varphi(\mathbf{x}_\delta) = \mathcal{V}(\mathbf{E}_\delta)$  as above. Then  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta) = \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{x}_\delta) = \mathcal{V}(\mathbf{x}_\delta^\varphi) = \mathcal{V}(\mathbf{E}_\delta) = \varphi(\mathbf{x}_\delta)$ .
2. Let  $\mathbf{D}_\delta$  be a primitive constant. Then  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta) = \mathcal{V}(\mathbf{D}_\delta^\varphi) = \mathcal{V}(\mathbf{D}_\delta) = \mathcal{J}(\mathbf{D}_\delta)$ .
3. Let  $\mathbf{D}_\delta$  be  $[\mathbf{A}_{\alpha\beta}\mathbf{B}_\beta]$ . If  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_{\alpha\beta})$  is defined,  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_\beta)$  is defined, and  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_{\alpha\beta})$  is defined at  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_\beta)$ , then  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta) = \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_{\alpha\beta}\mathbf{B}_\beta) = \mathcal{V}(\mathbf{A}_{\alpha\beta}^\varphi\mathbf{B}_\beta^\varphi) = \mathcal{V}(\mathbf{A}_{\alpha\beta}^\varphi)(\mathcal{V}(\mathbf{B}_\beta^\varphi)) = \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_{\alpha\beta})(\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_\beta))$ . Now assume  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_{\alpha\beta})$  is undefined,  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_\beta)$  is undefined, or  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_{\alpha\beta})$  is not defined at  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_\beta)$ . Then  $\mathcal{H} \vdash^{\text{ef}} \mathbf{A}_{\alpha\beta}^\varphi \uparrow$ ,  $\mathcal{H} \vdash^{\text{ef}} \mathbf{B}_\beta^\varphi \uparrow$ , or  $\mathcal{V}(\mathbf{A}_{\alpha\beta}^\varphi\mathbf{B}_\beta^\varphi)$  is undefined.  $\mathcal{H} \vdash^{\text{ef}} \mathbf{A}_{\alpha\beta}^\varphi \uparrow$  or  $\mathcal{H} \vdash^{\text{ef}} \mathbf{B}_\beta^\varphi \uparrow$  implies  $\mathcal{H} \vdash^{\text{ef}} \mathbf{A}_{\alpha\beta}^\varphi\mathbf{B}_\beta^\varphi \simeq \perp_\alpha$  by Axiom 6.4. If  $\alpha = o$ , then  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta) = \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_{\alpha\beta}\mathbf{B}_\beta) = \mathcal{V}(\mathbf{A}_{\alpha\beta}^\varphi\mathbf{B}_\beta^\varphi) = \mathcal{V}(F_o) = \mathbf{F}$ . If  $\alpha \neq o$ , then  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta) \simeq \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_{\alpha\beta}\mathbf{B}_\beta) \simeq \mathcal{V}(\mathbf{A}_{\alpha\beta}^\varphi\mathbf{B}_\beta^\varphi) \simeq \mathcal{V}(\perp_\alpha)$  is undefined by Axiom 6.11.
4. Let  $\mathbf{D}_\delta$  be  $[\lambda\mathbf{x}_\alpha\mathbf{B}_\beta]$ . Let  $\mathcal{V}(\mathbf{E}_\alpha)$  be an arbitrary member of  $\mathcal{D}_\alpha$ , and so  $\mathbf{E}_\alpha$  is an xwff and  $\mathcal{H} \vdash^{\text{ef}} \mathbf{E}_\alpha \downarrow$ . Given an assignment  $\varphi \in \text{assign}(\mathcal{M})$ , let  $\psi = \varphi[\mathbf{x}_\alpha \mapsto \mathcal{V}(\mathbf{E}_\alpha)]$ . It follows from the Beta-Reduction Theorem (part 1) that  $\mathcal{H} \vdash^{\text{ef}} [\lambda\mathbf{x}_\alpha\mathbf{B}_\beta]^\varphi\mathbf{E}_\alpha \simeq \mathbf{B}_\beta^\psi$ . Then  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta)(\mathcal{V}(\mathbf{E}_\alpha)) \simeq \mathcal{V}_\varphi^{\mathcal{M}}([\lambda\mathbf{x}_\alpha\mathbf{B}_\beta])(\mathcal{V}(\mathbf{E}_\alpha)) \simeq \mathcal{V}([\lambda\mathbf{x}_\alpha\mathbf{B}_\beta]^\varphi)(\mathcal{V}(\mathbf{E}_\alpha)) \simeq \mathcal{V}([\lambda\mathbf{x}_\alpha\mathbf{B}_\beta]^\varphi\mathbf{E}_\alpha) \simeq \mathcal{V}(\mathbf{B}_\beta^\psi) \simeq \mathcal{V}_\psi^{\mathcal{M}}(\mathbf{B}_\beta)$  as required.
5. Let  $\mathbf{D}_\delta$  be  $[\mathbf{cA}_o\mathbf{B}_\alpha\mathbf{C}_\alpha]$ . If  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_o) = \mathbf{T}$ , then  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta) \simeq \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{cA}_o\mathbf{B}_\alpha\mathbf{C}_\alpha) \simeq \mathcal{V}([\mathbf{cA}_o\mathbf{B}_\alpha\mathbf{C}_\alpha]^\varphi) \simeq \mathcal{V}(\mathbf{cA}_o^\varphi\mathbf{B}_\alpha^\varphi\mathbf{C}_\alpha^\varphi) \simeq \mathcal{V}(cT_o\mathbf{B}_\alpha^\varphi\mathbf{C}_\alpha^\varphi) \simeq \mathcal{V}(\mathbf{B}_\alpha^\varphi) \simeq \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_\alpha)$  by Axiom 10.1. Similarly, if  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_o) = \mathbf{F}$ , then  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta) \simeq \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_\alpha)$  by Axiom 10.2.
6. Let  $\mathbf{D}_\delta$  be  $[\mathbf{qA}_\alpha]$ . Then  $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{D}_\delta) = \mathcal{V}_\varphi^{\mathcal{M}}([\mathbf{qA}_\alpha]) = \mathcal{V}([\mathbf{qA}_\alpha]^\varphi) = \mathcal{V}([\mathbf{qA}_\alpha]) = \mathcal{E}(\mathbf{A}_\alpha)$ .

Thus  $\mathcal{M}$  is an evaluation-free model for  $\overline{\mathcal{Q}_0^{\text{uqe}}}$ .

**Step 4** We must show that  $\mathcal{M}$  is normal and frugal. If  $\mathbf{A}_o$  is an evaluation-free specifying axiom given by Specifications 1–9, then  $\mathcal{H} \vdash^{\text{ef}} \mathbf{A}_o$  by Axiom 12, so  $\mathcal{V}(\mathbf{A}_o) = \mathbf{T}$  and  $\mathcal{M} \models \mathbf{A}_o$ , and so  $\mathcal{M}$  is normal. Clearly, (a)  $\text{card}(\mathcal{D}_\alpha) \leq \text{card}(\mathcal{L}(\overline{\mathcal{Q}_0^{\text{uqe}}}))$  since  $\mathcal{V}$  maps a subset of the xwffs $_\alpha$  of  $\overline{\mathcal{Q}_0^{\text{uqe}}}$  onto  $\mathcal{D}_\alpha$  and (b)  $\text{card}(\mathcal{L}(\overline{\mathcal{Q}_0^{\text{uqe}}})) = \text{card}(\mathcal{L}(\mathcal{Q}_0^{\text{uqe}}))$ , and so  $\mathcal{M}$  is frugal.

**Step 5** We must show that  $\mathcal{M}$  is a frugal normal evaluation-free model for  $\mathcal{G}$ . We have shown that  $\mathcal{M}$  is a frugal normal evaluation-free model for  $\overline{\mathcal{Q}_0^{\text{uqe}}}$ . Clearly,  $\mathcal{M}$  is also a frugal normal evaluation-free model for  $\mathcal{Q}_0^{\text{uqe}}$ . If  $\mathbf{A}_o \in \mathcal{G}$ ,

then  $\mathbf{A}_o \in \mathcal{H}$ , so  $\mathcal{H} \vdash^{\text{ef}} \mathbf{A}_o$ , so  $\mathcal{V}(\mathbf{A}_o) = \top$  and  $\mathcal{M} \models \mathbf{A}_o$ , and so  $\mathcal{M}$  is an evaluation-free model for  $\mathcal{G}$ .  $\square$

### 11.3 Evaluation-Free Completeness Theorem

**Theorem 11.3.1 (Evaluation-Free Completeness Theorem for  $\mathcal{P}^{\text{uqe}}$ )**  
 $\mathcal{P}^{\text{uqe}}$  is evaluation-free complete for  $\mathcal{Q}_0^{\text{uqe}}$ .

**Proof** Let  $\mathbf{A}_o$  be an evaluation-free formula and  $\mathcal{H}$  be a set of syntactically closed evaluation-free formulas of  $\mathcal{Q}_0^{\text{uqe}}$ . Assume  $\mathcal{H} \models_n^{\text{ef}} \mathbf{A}_o$ , and let  $\mathbf{B}_o$  be a universal closure of  $\mathbf{A}_o$ . Then  $\mathbf{B}_o$  is syntactically closed and  $\mathcal{H} \models_n^{\text{ef}} \mathbf{B}_o$  by Lemma 7.2.3. Suppose  $\mathcal{H} \cup \{\sim \mathbf{B}_o\}$  is consistent in  $\mathcal{P}^{\text{uqe}}$ . Then, by Henkin's Theorem, there is a normal evaluation-free model  $\mathcal{M}$  for  $\mathcal{H} \cup \{\sim \mathbf{B}_o\}$ , and so  $\mathcal{M} \models \sim \mathbf{B}_o$ . Since  $\mathcal{M}$  is also a normal evaluation-free model for  $\mathcal{H}$ ,  $\mathcal{M} \models \mathbf{B}_o$ . From this contradiction it follows that  $\mathcal{H} \cup \{\sim \mathbf{B}_o\}$  is inconsistent in  $\mathcal{P}^{\text{uqe}}$ . Hence  $\mathcal{H} \vdash^{\text{ef}} \mathbf{B}_o$  by the Deduction Theorem and the Tautology Theorem. Therefore,  $\mathcal{H} \vdash^{\text{ef}} \mathbf{A}_o$  by Universal Instantiation (part 1) and Axiom 6.1.  $\square$

## 12 Applications

We will now look at some applications of the machinery in  $\mathcal{Q}_0^{\text{uqe}}$  for reasoning about the interplay of the syntax and semantics of  $\mathcal{Q}_0^{\text{uqe}}$  expressions (i.e., wffs). We will consider three kinds of applications. The first kind uses the type  $\epsilon$  machinery to reason about the syntactic structure of wffs; see the examples in subsection 12.1. The second kind uses evaluation applied to variables of type  $\epsilon$  to express syntactic variables as employed, for example, in schemas; see the examples in subsections 12.2 and 12.3. The third kind uses the full machinery of  $\mathcal{Q}_0^{\text{uqe}}$  to formalize syntax-based mathematical algorithms in the manner described in [25]; see the example in subsection 12.4.

### 12.1 Example: Implications

We will illustrate how the type  $\epsilon$  machinery in  $\mathcal{Q}_0^{\text{uqe}}$  can be used to reason about the syntactic structure of wffs by defining some useful constants for analyzing and manipulating *implications*, i.e., formulas of the form  $\mathbf{A}_o \supset \mathbf{B}_o$ .

Let  $\text{implies}_{\epsilon\epsilon\epsilon}$  be a defined constant that stands for

$$\lambda x_\epsilon \lambda y_\epsilon [\text{app}_{\epsilon\epsilon\epsilon} [\text{app}_{\epsilon\epsilon\epsilon} \ulcorner \supset_{ooo} \urcorner x_\epsilon ] y_\epsilon].$$

**Lemma 12.1.1** For all formulas  $\mathbf{A}_o$  and  $\mathbf{B}_o$ ,

$$\vdash \text{implies}_{\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_o \urcorner \ulcorner \mathbf{B}_o \urcorner = \ulcorner \mathbf{A}_o \supset \mathbf{B}_o \urcorner.$$

**Proof**

$$\vdash \text{implies}_{\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_o \urcorner \ulcorner \mathbf{B}_o \urcorner \simeq \text{implies}_{\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_o \urcorner \ulcorner \mathbf{B}_o \urcorner. \quad (1)$$

$$\begin{aligned} \vdash \text{implies}_{\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_o \urcorner \ulcorner \mathbf{B}_o \urcorner &\simeq \\ &[\lambda x_\epsilon \lambda y_\epsilon [\text{app}_{\epsilon\epsilon\epsilon} [\text{app}_{\epsilon\epsilon\epsilon} \ulcorner \supset_{ooo} \urcorner x_\epsilon] y_\epsilon]] \ulcorner \mathbf{A}_o \urcorner \ulcorner \mathbf{B}_o \urcorner. \end{aligned} \quad (2)$$

$$\vdash \text{implies}_{\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_o \urcorner \ulcorner \mathbf{B}_o \urcorner \simeq \text{app}_{\epsilon\epsilon\epsilon} [\text{app}_{\epsilon\epsilon\epsilon} \ulcorner \supset_{ooo} \urcorner \ulcorner \mathbf{A}_o \urcorner] \ulcorner \mathbf{B}_o \urcorner. \quad (3)$$

$$\vdash \text{implies}_{\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_o \urcorner \ulcorner \mathbf{B}_o \urcorner \simeq \text{app}_{\epsilon\epsilon\epsilon} \ulcorner \supset_{ooo} \urcorner \ulcorner \mathbf{A}_o \urcorner \ulcorner \mathbf{B}_o \urcorner. \quad (4)$$

$$\vdash \text{implies}_{\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_o \urcorner \ulcorner \mathbf{B}_o \urcorner = \ulcorner \supset_{ooo} \urcorner \ulcorner \mathbf{A}_o \mathbf{B}_o \urcorner. \quad (5)$$

$$\vdash \text{implies}_{\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_o \urcorner \ulcorner \mathbf{B}_o \urcorner = \ulcorner \mathbf{A}_o \supset \mathbf{B}_o \urcorner. \quad (6)$$

(1) is by Proposition 10.1.1; (2) follows from (1) by the definition of  $\text{implies}_{\epsilon\epsilon\epsilon}$ ; (3) follows from (2) and Axioms 4.2–5 and 6.6 by Rules 1 and 2; (4) follows from (3) by Specification 1; (5) follows from (4) by Specification 1; and (6) follows from (5) by abbreviation.  $\square$

That is,  $\text{implies}_{\epsilon\epsilon\epsilon}$  is an *implication constructor*: the application of it to the syntactic representations of two formulas  $\mathbf{A}_o$  and  $\mathbf{B}_o$  denotes the syntactic representation of the implication  $\mathbf{A}_o \supset \mathbf{B}_o$ .

Let  $\text{is-implication}_{o\epsilon}$  be a defined constant that stands for

$$\lambda x_\epsilon \exists y_\epsilon \exists z_\epsilon [x_\epsilon = \text{implies}_{\epsilon\epsilon\epsilon} y_\epsilon z_\epsilon].$$

That is,  $\text{is-implication}_{o\epsilon}$  is an *implication recognizer*: the application of it to the syntactic representation of a formula  $\mathbf{A}_o$  has the value  $\top$  iff  $\mathbf{A}_o$  has the form  $\mathbf{B}_o \supset \mathbf{C}_o$ .

Let  $\text{antecedent}_{\epsilon\epsilon}$  and  $\text{succedent}_{\epsilon\epsilon}$  be the defined constants that, respectively, stand for

$$\lambda x_\epsilon \text{I}y_\epsilon \exists z_\epsilon [x_\epsilon = \text{implies}_{\epsilon\epsilon\epsilon} y_\epsilon z_\epsilon]$$

and

$$\lambda x_\epsilon \text{I}z_\epsilon \exists y_\epsilon [x_\epsilon = \text{implies}_{\epsilon\epsilon\epsilon} y_\epsilon z_\epsilon].$$

Then

$$\vdash \text{antecedent}_{\epsilon\epsilon} \ulcorner \mathbf{A}_o \supset \mathbf{B}_o \urcorner = \ulcorner \mathbf{A}_o \urcorner$$

and

$$\vdash \text{succedent}_{\epsilon\epsilon} \ulcorner \mathbf{A}_o \supset \mathbf{B}_o \urcorner = \ulcorner \mathbf{B}_o \urcorner.$$

That is,  $\text{antecedent}_{\epsilon\epsilon}$  and  $\text{succedent}_{\epsilon\epsilon}$  are *implication deconstructors*: the applications of them to the syntactic representation of a formula  $\mathbf{A}_o$  denote the syntactic representations of the antecedent and succedent, respectively, of  $\mathbf{A}_o$  if  $\mathbf{A}_o$  is an implication and are undefined otherwise.

Let  $\text{converse}_{\epsilon\epsilon}$  be a defined constant that stands for

$$\lambda x_\epsilon [\text{implies}_{\epsilon\epsilon} [\text{succedent}_{\epsilon\epsilon} x_\epsilon] [\text{antecedent}_{\epsilon\epsilon} x_\epsilon]].$$

Then

$$\vdash \text{converse}_{\epsilon\epsilon} \ulcorner \mathbf{A}_o \supset \mathbf{B}_o \urcorner = \ulcorner \mathbf{B}_o \supset \mathbf{A}_o \urcorner.$$

That is,  $\text{converse}_{\epsilon\epsilon}$  is an *implication converser*: the application of it to the syntactic representation of a formula  $\mathbf{A}_o$  denotes the syntactic representation of the converse of  $\mathbf{A}_o$  if  $\mathbf{A}_o$  is an implication and is undefined otherwise.

## 12.2 Example: Law of Excluded Middle

The value of a wff of the form  $\llbracket \mathbf{x}_\epsilon \rrbracket_\alpha$  ranges over the values of wffs of type  $\alpha$ . Thus wffs like  $\llbracket \mathbf{x}_\epsilon \rrbracket_\alpha$  can be used in other wffs as syntactic variables. It is thus possible to express schemas as single wffs in  $\mathcal{Q}_0^{\text{uqe}}$ . As an example, let us consider the *law of excluded middle (LEM)* which is usually written as a formula schema like

$$\mathbf{A}_o \vee \sim \mathbf{A}_o$$

where  $\mathbf{A}_o$  ranges over all formulas. LEM can be naively represented in  $\mathcal{Q}_0^{\text{uqe}}$  as

$$\forall x_\epsilon [\llbracket x_\epsilon \rrbracket_o \vee \sim \llbracket x_\epsilon \rrbracket_o].$$

The variable  $x_\epsilon$  ranges over the syntactic representations of all wffs, not just formulas. However,  $\llbracket x_\epsilon \rrbracket_o$  is false when the value of  $x_\epsilon$  is not an evaluation-free formula. A more intensionally correct representation of LEM is

$$\forall x_\epsilon [\text{eval-free}_{o\epsilon}^o x_\epsilon \supset [\llbracket x_\epsilon \rrbracket_o \vee \sim \llbracket x_\epsilon \rrbracket_o]]$$

where  $x_\epsilon$  is restricted to the syntactic representations of evaluation-free formulas. This representation of LEM is a theorem of  $\mathcal{P}^{\text{uqe}}$ :

**Lemma 12.2.1**  $\vdash \forall x_\epsilon [\text{eval-free}_{o\epsilon}^o x_\epsilon \supset [\llbracket x_\epsilon \rrbracket_o \vee \sim \llbracket x_\epsilon \rrbracket_o]].$

**Proof**

$$\vdash x_o \vee \sim x_o. \tag{1}$$

$$\vdash \forall x_o [x_o \vee \sim x_o]. \tag{2}$$

$$\begin{aligned} \vdash [\lambda x_o [x_o \vee \sim x_o]] [\text{if} [\text{eval-free}_{o\epsilon}^o x_\epsilon] \llbracket x_\epsilon \rrbracket_o \perp_o] &\simeq \\ &[\text{if} [\text{eval-free}_{o\epsilon}^o x_\epsilon] \llbracket x_\epsilon \rrbracket_o \perp_o] \vee \sim [\text{if} [\text{eval-free}_{o\epsilon}^o x_\epsilon] \llbracket x_\epsilon \rrbracket_o \perp_o]. \end{aligned} \tag{3}$$

$$\vdash [\text{if} [\text{eval-free}_{o\epsilon}^o x_\epsilon] \llbracket x_\epsilon \rrbracket_o \perp_o] \vee \sim [\text{if} [\text{eval-free}_{o\epsilon}^o x_\epsilon] \llbracket x_\epsilon \rrbracket_o \perp_o]. \tag{4}$$

$$\vdash \text{eval-free}_{o\epsilon}^o x_\epsilon \supset [\llbracket x_\epsilon \rrbracket_o \vee \sim \llbracket x_\epsilon \rrbracket_o]. \tag{5}$$

$$\vdash \forall x_\epsilon [\text{eval-free}_{o\epsilon}^o x_\epsilon \supset [\llbracket x_\epsilon \rrbracket_o \vee \sim \llbracket x_\epsilon \rrbracket_o]]. \tag{6}$$

(1) is by Axiom 5; (2) follows from (1) by Universal Generalization; (3) follows from Axioms 4.2–4 and Proposition 10.1.6 by Rules 1 and 2; (4) follows from (2) and (3) by Universal Instantiation (part 3); (5) follows from (4) by the Axiom 10 and the Tautology Theorem; and (6) follows from (5) by Universal Generalization.  $\square$

### 12.3 Example: Law of Beta-Reduction

Axiom 4.1, the law of beta-reduction for  $\mathcal{Q}_0^{\text{uqe}}$ , can be expressed as the following schema whose only syntactic variables are  $\alpha$  and  $\beta$ :

$$\begin{aligned} \forall x_\epsilon \forall y_\epsilon \forall z_\epsilon \forall z'_\epsilon [ [x_\epsilon]_\alpha \downarrow \wedge [\text{var}_{o_\epsilon}^\alpha y_\epsilon] \wedge [\text{wff}_{o_\epsilon}^\beta z_\epsilon] \wedge [\text{wff}_{o_\epsilon}^\beta z'_\epsilon] \wedge \\ \text{sub}_{\epsilon\epsilon\epsilon\epsilon} x_\epsilon y_\epsilon z_\epsilon = z'_\epsilon ] \supset [ \text{app}_{\epsilon\epsilon\epsilon} [\text{abs}_{\epsilon\epsilon\epsilon} y_\epsilon z_\epsilon] x_\epsilon ]_\beta \simeq [z'_\epsilon]_\beta \end{aligned}$$

Each instance of this schema (for a chosen  $\alpha$  and  $\beta$ ) is valid in  $\mathcal{Q}_0^{\text{uqe}}$  but not provable in  $\mathcal{P}^{\text{uqe}}$ . Moreover, the instances of an instance  $\mathbf{A}_o$  of this schema are not provable in  $\mathcal{P}^{\text{uqe}}$  from  $\mathbf{A}_o$  since  $\mathbf{A}_o$  contains the evaluation

$$[ \text{app}_{\epsilon\epsilon\epsilon} [\text{abs}_{\epsilon\epsilon\epsilon} y_\epsilon z_\epsilon] x_\epsilon ]_\beta$$

in which more than one variable is free.

Using the technique of grouping variables together described in Example 1 in subsection 7.8, we can also express Axiom 4.1 as the following schema that contains just the single variable  $x_{\langle\epsilon\epsilon\rangle\langle\epsilon\epsilon\rangle}$ :

$$\begin{aligned} \forall x_{\langle\epsilon\epsilon\rangle\langle\epsilon\epsilon\rangle} [ [ [ \mathbf{X}_\epsilon ]_\alpha \downarrow \wedge [\text{var}_{o_\epsilon}^\alpha \mathbf{Y}_\epsilon] \wedge [\text{wff}_{o_\epsilon}^\beta \mathbf{Z}_\epsilon] \wedge [\text{wff}_{o_\epsilon}^\beta \mathbf{Z}'_\epsilon] \wedge \\ \text{sub}_{\epsilon\epsilon\epsilon\epsilon} \mathbf{X}_\epsilon \mathbf{Y}_\epsilon \mathbf{Z}_\epsilon = \mathbf{Z}'_\epsilon ] \supset [ \text{app}_{\epsilon\epsilon\epsilon} [\text{abs}_{\epsilon\epsilon\epsilon} \mathbf{Y}_\epsilon \mathbf{Z}_\epsilon] \mathbf{X}_\epsilon ]_\beta \simeq [ \mathbf{Z}'_\epsilon ]_\beta \end{aligned}$$

where:

$$\begin{aligned} \mathbf{X}_\epsilon &\text{ is } \text{fst}_{\epsilon\langle\epsilon\epsilon\rangle} [ \text{fst}_{\langle\epsilon\epsilon\rangle\langle\epsilon\epsilon\rangle\langle\epsilon\epsilon\rangle} x_{\langle\epsilon\epsilon\rangle\langle\epsilon\epsilon\rangle} ] \\ \mathbf{Y}_\epsilon &\text{ is } \text{snd}_{\epsilon\langle\epsilon\epsilon\rangle} [ \text{fst}_{\langle\epsilon\epsilon\rangle\langle\epsilon\epsilon\rangle\langle\epsilon\epsilon\rangle} x_{\langle\epsilon\epsilon\rangle\langle\epsilon\epsilon\rangle} ] \\ \mathbf{Z}_\epsilon &\text{ is } \text{fst}_{\epsilon\langle\epsilon\epsilon\rangle} [ \text{snd}_{\langle\epsilon\epsilon\rangle\langle\epsilon\epsilon\rangle\langle\epsilon\epsilon\rangle} x_{\langle\epsilon\epsilon\rangle\langle\epsilon\epsilon\rangle} ] \\ \mathbf{Z}'_\epsilon &\text{ is } \text{snd}_{\epsilon\langle\epsilon\epsilon\rangle} [ \text{snd}_{\langle\epsilon\epsilon\rangle\langle\epsilon\epsilon\rangle\langle\epsilon\epsilon\rangle} x_{\langle\epsilon\epsilon\rangle\langle\epsilon\epsilon\rangle} ]. \end{aligned}$$

Like the first schema, each instance of this second schema is valid in  $\mathcal{Q}_0^{\text{uqe}}$  but not provable in  $\mathcal{P}^{\text{uqe}}$ . However, unlike the first schema, the instances of an instance  $\mathbf{A}_o$  of the second schema are provable in  $\mathcal{P}^{\text{uqe}}$  from  $\mathbf{A}_o$ .

### 12.4 Example: Conjunction Construction

Suppose  $A$  is an algorithm that, given two formulas  $\mathbf{A}_o$  and  $\mathbf{B}_o$  as input, returns as output (1)  $\mathbf{B}_o$  if  $\mathbf{A}_o$  is  $T_o$ , (2)  $\mathbf{A}_o$  if  $\mathbf{B}_o$  is  $T_o$ , (3)  $F_o$  if either  $\mathbf{A}_o$  or  $\mathbf{B}_o$  is  $F_o$ , or (4)  $\mathbf{A}_o \wedge \mathbf{B}_o$  otherwise. Although this is a trivial algorithm, we can use it to illustrate how a syntax-based mathematical algorithm can be formalized in  $\mathcal{Q}_0^{\text{uqe}}$ . As described in [25] we need to do the following three things to formalize  $A$  in  $\mathcal{Q}_0^{\text{uqe}}$ .

1. Define an operator  $O_A$  in  $\mathcal{Q}_0^{\text{uqe}}$  as a constant that represents  $A$ .
2. Prove in  $\mathcal{P}^{\text{uqe}}$  that  $O_A$  is mathematically correct.
3. Devise a mechanism for using  $O_A$  in  $\mathcal{Q}_0^{\text{uqe}}$ .



Let  $\mathbf{and}_{\epsilon\epsilon\epsilon}$  be a defined constant that stands for

$$\lambda x_\epsilon \lambda y_\epsilon [\mathbf{app}_{\epsilon\epsilon\epsilon} [\mathbf{app}_{\epsilon\epsilon\epsilon} \ulcorner \wedge_{ooo} \urcorner x_\epsilon] y_\epsilon].$$

Then

$$\vdash \mathbf{and}_{\epsilon\epsilon\epsilon} \ulcorner \mathbf{A}_o \urcorner \ulcorner \mathbf{B}_o \urcorner = \ulcorner \mathbf{A}_o \wedge \mathbf{B}_o \urcorner$$

for all formulas  $\mathbf{A}_o$  and  $\mathbf{B}_o$  as shown by a derivation similar to the one for  $\mathbf{implies}_{\epsilon\epsilon\epsilon}$  in the proof of Lemma 12.1.1. Define  $O_A$  to be  $\mathbf{and}\text{-simp}_{\epsilon\epsilon\epsilon}$ , a defined constant that stands for

$$\begin{aligned} & \lambda x_\epsilon \lambda y_\epsilon [\text{if } [x_\epsilon = \ulcorner T_o \urcorner] y_\epsilon \\ & \quad [\text{if } [y_\epsilon = \ulcorner T_o \urcorner] x_\epsilon \\ & \quad \quad [\text{if } [x_\epsilon = \ulcorner F_o \urcorner] \ulcorner F_o \urcorner \\ & \quad \quad \quad [\text{if } [y_\epsilon = \ulcorner F_o \urcorner] \ulcorner F_o \urcorner \\ & \quad \quad \quad \quad [\mathbf{and}_{\epsilon\epsilon\epsilon} x_\epsilon y_\epsilon]]]]]. \end{aligned}$$

The sentence

$$\begin{aligned} & \forall x_\epsilon \forall y_\epsilon [[\mathbf{wff}_{o\epsilon}^o x_\epsilon \wedge \mathbf{wff}_{o\epsilon}^o y_\epsilon] \supset \\ & \quad [[x_\epsilon = \ulcorner T_o \urcorner \supset \mathbf{and}\text{-simp}_{\epsilon\epsilon\epsilon} x_\epsilon y_\epsilon = y_\epsilon] \wedge \\ & \quad [y_\epsilon = \ulcorner T_o \urcorner \supset \mathbf{and}\text{-simp}_{\epsilon\epsilon\epsilon} x_\epsilon y_\epsilon = x_\epsilon] \wedge \\ & \quad [[x_\epsilon = \ulcorner F_o \urcorner \vee y_\epsilon = \ulcorner F_o \urcorner] \supset \mathbf{and}\text{-simp}_{\epsilon\epsilon\epsilon} x_\epsilon y_\epsilon = \ulcorner F_o \urcorner] \wedge \\ & \quad [[x_\epsilon \neq \ulcorner T_o \urcorner \wedge y_\epsilon \neq \ulcorner T_o \urcorner \wedge x_\epsilon \neq \ulcorner F_o \urcorner \wedge y_\epsilon \neq \ulcorner F_o \urcorner] \supset \\ & \quad \quad \mathbf{and}\text{-simp}_{\epsilon\epsilon\epsilon} x_\epsilon y_\epsilon = \mathbf{and}_{\epsilon\epsilon\epsilon} x_\epsilon y_\epsilon]], \end{aligned}$$

called  $\mathbf{CompBehavior}$ , specifies the intended *computational behavior* of  $O_A$ .

**Theorem 12.4.1 (Computational Behavior of  $\mathbf{and}\text{-simp}_{\epsilon\epsilon\epsilon}$ )**

$$\vdash \mathbf{CompBehavior}.$$

**Proof**  $\mathbf{CompBehavior}$  follows easily in  $\mathcal{P}^{\text{uqe}}$  from the definition of  $\mathbf{and}\text{-simp}_{\epsilon\epsilon\epsilon}$ .  $\square$

Hence  $O_A$  represents  $A$  by virtue of having the same computational behavior as that of  $A$ .

Let us make the following definitions:

$$\mathbf{P}_o \text{ is } [[\mathbf{and}_{\epsilon\epsilon\epsilon} [\mathbf{fst}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}] [\mathbf{snd}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}]]]_o.$$

$$\mathbf{Q}_o \text{ is } [[\mathbf{and}\text{-simp}_{\epsilon\epsilon\epsilon} [\mathbf{fst}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}] [\mathbf{snd}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}]]]_o.$$

$$\mathbf{R}_o \text{ is } [[\mathbf{fst}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}]_o \wedge [[\mathbf{snd}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}]]_o.$$

$$\begin{aligned}
\mathbf{S}_o \text{ is } & \text{[if [[fst}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}] = \ulcorner T_o \urcorner \llbracket \text{snd}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle} \rrbracket_o \\
& \text{[if [[snd}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}] = \ulcorner T_o \urcorner \llbracket \text{fst}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle} \rrbracket_o \\
& \text{[if [[fst}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}] = \ulcorner F_o \urcorner \llbracket \ulcorner F_o \urcorner \rrbracket_o \\
& \text{[if [[snd}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}] = \ulcorner F_o \urcorner \llbracket \ulcorner F_o \urcorner \rrbracket_o \\
& \mathbf{P}_o \rrbracket \rrbracket \rrbracket].
\end{aligned}$$

The formula

$$\forall x_{\langle\epsilon\epsilon\rangle} \llbracket \text{wff}_{o\epsilon}^o [\text{fst}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}] \wedge \text{wff}_{o\epsilon}^o [\text{snd}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}] \rrbracket \supset [\mathbf{Q}_o \equiv \mathbf{R}_o],$$

called **MathMeaning**, expresses the intended *mathematical meaning* of  $O_A$ . We will show that **MathMeaning** is a theorem of  $\mathcal{P}^{\text{uqe}}$  via a series of lemmas.

The first lemma asserts that the analog of the **MathMeaning** for  $\text{and}_{\epsilon\epsilon\epsilon}$  is a theorem of  $\mathcal{P}^{\text{uqe}}$ :

$$\mathbf{Lemma 12.4.2} \vdash \forall x_{\langle\epsilon\epsilon\rangle} \llbracket \text{wff}_{o\epsilon}^o [\text{fst}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}] \wedge \text{wff}_{o\epsilon}^o [\text{snd}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}] \rrbracket \supset [\mathbf{P}_o \equiv \mathbf{R}_o].$$

**Proof**

$$\vdash \llbracket \text{wff}_{o\epsilon}^o [\text{fst}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}] \wedge \text{wff}_{o\epsilon}^o [\text{snd}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}] \rrbracket \supset \mathbf{P}_o \equiv \mathbf{P}_o \quad (1)$$

$$\begin{aligned} & \vdash \llbracket \text{wff}_{o\epsilon}^o [\text{fst}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}] \wedge \text{wff}_{o\epsilon}^o [\text{snd}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}] \rrbracket \supset \\ & \mathbf{P}_o \equiv \llbracket \text{app}_{\epsilon\epsilon\epsilon} [\text{app}_{\epsilon\epsilon\epsilon} \ulcorner \wedge_{ooo} \urcorner [\text{fst}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}]] [\text{snd}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}] \rrbracket_o. \end{aligned} \quad (2)$$

$$\begin{aligned} & \vdash \llbracket \text{wff}_{o\epsilon}^o [\text{fst}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}] \wedge \text{wff}_{o\epsilon}^o [\text{snd}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}] \rrbracket \supset \\ & \mathbf{P}_o \equiv \llbracket \llbracket \text{app}_{\epsilon\epsilon\epsilon} \ulcorner \wedge_{ooo} \urcorner [\text{fst}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}] \rrbracket_{oo} \llbracket \text{snd}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle} \rrbracket_o. \end{aligned} \quad (3)$$

$$\begin{aligned} & \vdash \llbracket \text{wff}_{o\epsilon}^o [\text{fst}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}] \wedge \text{wff}_{o\epsilon}^o [\text{snd}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}] \rrbracket \supset \\ & \mathbf{P}_o \equiv \llbracket \ulcorner \wedge_{ooo} \urcorner \rrbracket_{ooo} \llbracket \text{fst}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle} \rrbracket_o \llbracket \text{snd}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle} \rrbracket_o. \end{aligned} \quad (4)$$

$$\begin{aligned} & \vdash \llbracket \text{wff}_{o\epsilon}^o [\text{fst}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}] \wedge \text{wff}_{o\epsilon}^o [\text{snd}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}] \rrbracket \supset \\ & \mathbf{P}_o \equiv \wedge_{ooo} \llbracket \text{fst}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle} \rrbracket_o \llbracket \text{snd}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle} \rrbracket_o. \end{aligned} \quad (5)$$

$$\vdash \llbracket \text{wff}_{o\epsilon}^o [\text{fst}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}] \wedge \text{wff}_{o\epsilon}^o [\text{snd}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}] \rrbracket \supset \mathbf{P}_o \equiv \mathbf{R}_o. \quad (6)$$

$$\vdash \forall x_{\langle\epsilon\epsilon\rangle} \llbracket \llbracket \text{wff}_{o\epsilon}^o [\text{fst}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}] \wedge \text{wff}_{o\epsilon}^o [\text{snd}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}] \rrbracket \rrbracket \supset [\mathbf{P}_o \equiv \mathbf{R}_o]. \quad (7)$$

(1) is by Lemmas 10.1.4, Propositions 10.1.1 and 10.1.6, and the Tautology Theorem; (2) follows from (1) by the definition of  $\text{and}_{\epsilon\epsilon\epsilon}$ , Axioms 4.2–5, and Lemma 10.2.1; (3) and (4) follow from (2) and (3), respectively, by Axiom 11.3, Lemma 10.2.1, and Specification 6; (5) follows from (4) by Axiom 11.2; (6) is by abbreviation; and (7) is by Universal Generalization.  $\square$

The second lemma shows how  $\mathbf{Q}_o$  can be reduced:

$$\mathbf{Lemma 12.4.3} \vdash \mathbf{Q}_o \equiv \mathbf{S}_o.$$

**Proof** The right side of the equation is obtained from the left side in three steps. First,  $\text{and}_{\text{simp}_{\epsilon\epsilon\epsilon}}$  is replaced by its definition. Second, the resulting formula is beta-reduced using Axioms 4.2–5 and 4.8 and parts 2 and 3 of Lemma 10.2.1. And third, evaluations are pushed inward using Axiom 10.3.  $\square$

The next lemma consists of five theorems of  $\mathcal{P}^{\text{uqe}}$ :

**Lemma 12.4.4**

1.  $\vdash x_{\langle\epsilon\epsilon\rangle} = [\text{pair}_{\langle oo \rangle oo} \ulcorner T_o \urcorner \urcorner T_o \urcorner] \supset [\mathbf{Q}_o \equiv \mathbf{R}_o]$ .
2.  $\vdash x_{\langle\epsilon\epsilon\rangle} = [\text{pair}_{\langle oo \rangle oo} \ulcorner F_o \urcorner \urcorner T_o \urcorner] \supset [\mathbf{Q}_o \equiv \mathbf{R}_o]$ .
3.  $\vdash x_{\langle\epsilon\epsilon\rangle} = [\text{pair}_{\langle oo \rangle oo} \ulcorner T_o \urcorner \urcorner F_o \urcorner] \supset [\mathbf{Q}_o \equiv \mathbf{R}_o]$ .
4.  $\vdash x_{\langle\epsilon\epsilon\rangle} = [\text{pair}_{\langle oo \rangle oo} \ulcorner F_o \urcorner \urcorner F_o \urcorner] \supset [\mathbf{Q}_o \equiv \mathbf{R}_o]$ .
5.  $\vdash [x_{\langle\epsilon\epsilon\rangle} \neq [\text{pair}_{\langle oo \rangle oo} \ulcorner T_o \urcorner \urcorner T_o \urcorner] \wedge x_{\langle\epsilon\epsilon\rangle} \neq [\text{pair}_{\langle oo \rangle oo} \ulcorner F_o \urcorner \urcorner T_o \urcorner] \wedge$   
 $x_{\langle\epsilon\epsilon\rangle} \neq [\text{pair}_{\langle oo \rangle oo} \ulcorner T_o \urcorner \urcorner F_o \urcorner] \wedge x_{\langle\epsilon\epsilon\rangle} \neq [\text{pair}_{\langle oo \rangle oo} \ulcorner F_o \urcorner \urcorner F_o \urcorner] \wedge$   
 $[\text{wff}_{o\epsilon}^o [\text{fst}_{\epsilon\langle\epsilon\epsilon\rangle} x_{\langle\epsilon\epsilon\rangle}] \wedge \text{wff}_{o\epsilon}^o [\text{snd}_{\epsilon\langle\epsilon\epsilon\rangle} x_{\langle\epsilon\epsilon\rangle}]]] \supset [\mathbf{Q}_o \equiv \mathbf{R}_o]$ .

**Proof**

**Part 1**

$$\vdash x_{\langle\epsilon\epsilon\rangle} = [\text{pair}_{\langle oo \rangle oo} \ulcorner T_o \urcorner \urcorner T_o \urcorner] \supset$$

$$[[\lambda x_{\langle\epsilon\epsilon\rangle} [\mathbf{Q}_o \equiv \mathbf{R}_o]] x_{\langle\epsilon\epsilon\rangle} \equiv [\lambda x_{\langle\epsilon\epsilon\rangle} [\mathbf{Q}_o \equiv \mathbf{R}_o]] [\text{pair}_{\langle oo \rangle oo} \ulcorner T_o \urcorner \urcorner T_o \urcorner]]. \quad (1)$$

$$\vdash x_{\langle\epsilon\epsilon\rangle} = [\text{pair}_{\langle oo \rangle oo} \ulcorner T_o \urcorner \urcorner T_o \urcorner] \supset$$

$$[[\mathbf{Q}_o \equiv \mathbf{R}_o] \equiv [\lambda x_{\langle\epsilon\epsilon\rangle} [\mathbf{Q}_o \equiv \mathbf{R}_o]] [\text{pair}_{\langle oo \rangle oo} \ulcorner T_o \urcorner \urcorner T_o \urcorner]]. \quad (2)$$

$$\vdash x_{\langle\epsilon\epsilon\rangle} = [\text{pair}_{\langle oo \rangle oo} \ulcorner T_o \urcorner \urcorner T_o \urcorner] \supset$$

$$[[\mathbf{Q}_o \equiv \mathbf{R}_o] \equiv [\lambda x_{\langle\epsilon\epsilon\rangle} [\mathbf{S}_o \equiv \mathbf{R}_o]] [\text{pair}_{\langle oo \rangle oo} \ulcorner T_o \urcorner \urcorner T_o \urcorner]]. \quad (3)$$

$$\vdash x_{\langle\epsilon\epsilon\rangle} = [\text{pair}_{\langle oo \rangle oo} \ulcorner T_o \urcorner \urcorner T_o \urcorner] \supset$$

$$[[\mathbf{Q}_o \equiv \mathbf{R}_o] \equiv [[\ulcorner T_o \urcorner]_o \equiv [\ulcorner T_o \urcorner]_o \wedge [[\ulcorner T_o \urcorner]_o]]]. \quad (4)$$

$$\vdash x_{\langle\epsilon\epsilon\rangle} = [\text{pair}_{\langle oo \rangle oo} \ulcorner T_o \urcorner \urcorner T_o \urcorner] \supset$$

$$[[\mathbf{Q}_o \equiv \mathbf{R}_o] \equiv T_o]. \quad (5)$$

$$\vdash x_{\langle\epsilon\epsilon\rangle} = [\text{pair}_{\langle oo \rangle oo} \ulcorner T_o \urcorner \urcorner T_o \urcorner] \supset [\mathbf{Q}_o \equiv \mathbf{R}_o]. \quad (6)$$

(1) is by Axiom 2; (2) follows from (1) by Axiom 4.10; (3) follows from (2) by Lemma 12.4.3 and Rule 1; (4) follows from (3) by Lemma 10.2.1, part 2 of the Beta-Reduction Theorem, Axiom 10.1, and the Tautology Theorem; (5) follows (4) by Axiom 11.2 and the Tautology Theorem; (6) follows from (5) by the Tautology Theorem.

**Part 2** Similar to Part 1.

**Part 3** Similar to Part 1.

**Part 4** Similar to Part 1.

**Part 5** Let  $\mathbf{A}_o$  be the antecedent of the implication in part 5 of the lemma.

$$\vdash \mathbf{P}_o \equiv \mathbf{R}_o \quad (1)$$

$$\vdash \mathbf{A}_o \supset [\mathbf{P}_o \equiv \mathbf{R}_o] \quad (2)$$

$$\vdash \mathbf{A}_o \supset [\mathbf{S}_o \equiv \mathbf{R}_o] \quad (3)$$

$$\vdash \mathbf{A}_o \supset [\mathbf{Q}_o \equiv \mathbf{R}_o] \quad (4)$$

(1) is by Lemma 12.4.2 and part 4 of Universal Instantiation; (2) follows from (1) by the Tautology Theorem; (3) follows from (2) by Axiom 10.2, Lemma 10.2.1, and the Tautology Theorem; and (4) follows from (3) by Lemma 12.4.3 and Rule 1.  $\square$

Finally, the theorem below shows that `MathMeaning` is a theorem of  $\mathcal{P}^{\text{uqe}}$ :

**Theorem 12.4.5 (Mathematical Meaning of `and-simp` <sub>$\epsilon\epsilon\epsilon$</sub> )**

$\vdash \text{MathMeaning}.$

**Proof**

$$\vdash [\text{wff}_{o\epsilon}^o [\text{fst}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}] \wedge \text{wff}_{o\epsilon}^o [\text{snd}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}]] \supset [\mathbf{Q}_o \equiv \mathbf{R}_o]$$

follows from Lemmas 10.2.1 and 12.4.4 and the Tautology Theorem. Then

$$\vdash \forall x_{\langle\epsilon\epsilon\rangle} [[\text{wff}_{o\epsilon}^o [\text{fst}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}] \wedge \text{wff}_{o\epsilon}^o [\text{snd}_{\epsilon(\epsilon\epsilon)} x_{\langle\epsilon\epsilon\rangle}]]] \supset [\mathbf{Q}_o \equiv \mathbf{R}_o]$$

follows from this by Universal Generalization.  $\square$

Hence  $O_A$  is mathematically correct.

While  $A$  manipulates formulas, `and-simp` <sub>$\epsilon\epsilon\epsilon$</sub>  manipulates syntactic representations of formulas. An application of  $O_A$  has the form `and-simp` <sub>$\epsilon\epsilon\epsilon$</sub>   $\ulcorner \mathbf{A}_o \urcorner \ulcorner \mathbf{B}_o \urcorner$ . Its value can be computed by expanding its definition, beta-reducing using Axiom 4, and then rewriting the resulting wff using Axiom 10 and Specification 1. If  $\mathbf{A}_o$  and  $\mathbf{B}_o$  are evaluation-free, its meaning can be obtained by instantiating the universal formula `MathMeaning` with the wff  $\langle \ulcorner \mathbf{A}_o \urcorner, \ulcorner \mathbf{B}_o \urcorner \rangle$  and then simplifying.

## 13 Conclusion

### 13.1 Summary of Results

We have presented a version of simple type theory called  $\mathcal{Q}_0^{\text{uqe}}$  that admits undefined expressions, quotations, and evaluations.  $\mathcal{Q}_0^{\text{uqe}}$  is based on  $\mathcal{Q}_0$ , a version of Church's type theory [12] developed by Peter B. Andrews [2].  $\mathcal{Q}_0^{\text{uqe}}$  directly formalizes the traditional approach to undefinedness [19] in which undefined expressions are treated as legitimate, nondenoting expressions that can be components of meaningful statements. It has the same facility for reasoning about undefinedness as  $\mathcal{Q}_0^{\text{u}}$  [23] that is derived from  $\mathcal{Q}_0$ . In addition, it has a facility for reasoning about the syntax of expressions based on quotation and evaluation.

The syntax of  $\mathcal{Q}_0^{\text{uqe}}$  differs from the syntax of  $\mathcal{Q}_0$  by having the following new machinery: a base type  $\epsilon$  that denotes a domain of syntactic values, a quotation operator, an evaluation operator, and several constants involving the type  $\epsilon$ .  $\mathcal{Q}_0^{\text{uqe}}$  also has some additional new machinery for ordered pairs and conditionals: a type constructor for forming types that denote domains of ordered

pairs, a constant for forming ordered pairs, and an expression constructor for forming conditionals. The semantics of  $\mathcal{Q}_0^{\text{uqe}}$  is based on Henkin-style general models [41] that include partial functions as well as total functions and in which expressions may be undefined. The expression constructor for conditionals is nonstrict with respect to undefinedness. An application of the quotation operator to an expression denotes a syntactic value that represents the expression. An application of the evaluation operator to an expression  $E$  denotes the value of the expression represented by the value of  $E$ . To avoid the Evaluation Problem mentioned in the Introduction, an evaluation  $\llbracket \ulcorner \mathbf{A}_\epsilon \urcorner \rrbracket_\alpha$  is undefined when  $\mathbf{A}_\epsilon$  is not evaluation-free.

The syntax and semantics of  $\mathcal{Q}_0^{\text{uqe}}$  are modest modifications of the syntax and semantics of  $\mathcal{Q}_0^{\text{u}}$ , but  $\mathcal{P}^{\text{uqe}}$ , the proof system of  $\mathcal{Q}_0^{\text{uqe}}$ , is a major modification of  $\mathcal{P}^{\text{u}}$ , the proof system of  $\mathcal{Q}_0^{\text{u}}$ . The substitution operation that is needed to perform beta-reduction is defined in the metalogic of  $\mathcal{Q}_0^{\text{u}}$ , while it is represented in  $\mathcal{Q}_0^{\text{uqe}}$  by a primitive constant  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$ . To avoid the Variable Problem mentioned in the Introduction,  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$  defines a semantics-dependent form of substitution. Moreover, the syntactic side conditions concerning free variables and substitution that are expressed in the metalogic of a traditional logic are expressed in the language of  $\mathcal{Q}_0^{\text{uqe}}$ . We prove that  $\mathcal{P}^{\text{uqe}}$  is sound with respect to the semantics of  $\mathcal{Q}_0^{\text{uqe}}$  (Theorem 9.2.1), but it is not complete. However, it is complete for evaluation-free formulas (Theorem 11.3.1).

$\mathcal{Q}_0^{\text{uqe}}$  is not complete because it is not possible to beta-reduce all applications of function abstraction. There are two ways of performing beta-reduction in  $\mathcal{Q}_0^{\text{uqe}}$ . The first way uses the specifying axioms of the primitive constant  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$  to perform substitution as expressed by Axiom 4.1. This first way works for all applications of function abstraction involving just evaluation-free wffs, but it works for only some applications involving evaluations. The second way uses the basic properties of lambda-notation as expressed by Axioms 4.2–10. Like the first way, this second way works for all applications of function abstraction involving just evaluation-free wffs, but it works for only some applications involving evaluations. However, the two ways complement each other because they work for different applications of function abstraction involving evaluations.

## 13.2 Significance of Results

The construction of  $\mathcal{Q}_0^{\text{uqe}}$  demonstrates how the replete approach to reasoning about syntax [25] — in which it is possible to reason about the syntax of the entire language of the logic using quotation and evaluation operators defined in the logic — can be implemented in Church’s type theory [12]. Moreover, the implementation ideas employed in  $\mathcal{Q}_0^{\text{uqe}}$  can be applied to other traditional logics like first-order logic. Even though the proof system of  $\mathcal{Q}_0^{\text{uqe}}$  is not complete, it is powerful enough to be useful. We have illustrated how  $\mathcal{Q}_0^{\text{uqe}}$  can be used to (1) reason about the syntactic structure of expressions, (2) represent and instantiate schemas with syntactic variables, and (3) formalize syntax-based mathematical algorithms in the sense given in [25]. We believe  $\mathcal{Q}_0^{\text{uqe}}$  is the first

implementation of the replete approach in a traditional logic.

The most innovative and complex part of  $\mathcal{Q}_0^{\text{uqe}}$  is the semantics-based form of substitution represented by the primitive constant  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$ . It provides the means to instantiate both variables occurring in evaluations and variables resulting from evaluations. In particular, it enables schemas expressed using evaluation (e.g., as given in subsections 12.2 and 12.3) to be instantiated. We showed that the substitution mechanism is correct by proving the law of beta-reduction formulated using  $\text{sub}_{\epsilon\epsilon\epsilon\epsilon}$  (Theorem 7.5.2). The proof of this theorem is intricate and involves many lemmas.

$\mathcal{Q}_0^{\text{uqe}}$  is intended primarily for theoretical purposes; it is not designed to be used in practice. A more practical version of  $\mathcal{Q}_0^{\text{uqe}}$  could be obtained by extending it in some of the ways discussed in [24]. For instance,  $\mathcal{Q}_0^{\text{uqe}}$  could be extended to include type variables as in the logic of the HOL theorem proving system [34] and its successors [40, 44, 51] or subtypes as in the logic of the IMPS theorem proving system [26, 27]. These additions would significantly raise the practical expressivity of the logic but would further raise the complexity of the logic. Many of these kinds of practical measures are implemented together in the logic Chiron [21, 22], a derivative of von-Neumann-Bernays-Gödel (NBG) set theory that admits undefined expressions, has a rich type system, and is equipped with a facility of reasoning about syntax that is very similar to  $\mathcal{Q}_0^{\text{uqe}}$ 's.

### 13.3 Related Work

#### Reasoning in Logic about Syntax

Reasoning in a logic about syntax begins with Kurt Gödel's famous use of *Gödel numbers* in [33] to encode expressions. Gödel, Tarski, and others used reasoning about syntax to show some of the limits of formal logic by *reflecting* the metalogic of a logic into the logic itself. *Reflection* is a technique to embed reasoning about a reasoning system (i.e., metareasoning) in the reasoning system itself. It very often involves the syntactic manipulation of expressions. Reflection has been employed in logic both for theoretical purposes [42] and practical purposes [39].

The technique of *deep embedding* is used to reason in a logic about the syntax of a particular language [8, 13, 67]. This is usually done with the local approach but could also be done with the global approach. A deep embedding can also provide a basis for formalizing syntax-based mathematical algorithms. Examples include the ring tactic implemented in Coq [14] and Wojciech Jedynek's semiring solver in Agda [49, 50, 66].

Florian Rabe proposes in [56] a method for freely adding literals for the values in a given semantic domain. This method can be used for reasoning about syntax by choosing a language of expressions as the semantic domain. Rabe's approach provides a quotation operation that is more general than the quotation operation we have defined for  $\mathcal{Q}_0^{\text{uqe}}$ . However, his approach does not provide an escape from obstacles like the Evaluation Problem and the Variable Problem described in section 1.

## Reasoning in the Lambda Calculus about Syntax

Corrado Böhm and Alessandro Berarducci present in [6] a method for representing an inductive type of values as a collection of lambda-terms. Then functions defined on the members of the inductive type can also be represented as lambda terms. Both the lambda terms representing the values and those representing the functions defined on the values can be typed in the second-order lambda calculus (System F) [31, 57] as shown in [6]. C. Böhm and his collaborators present in [5, 7] a second, more powerful method for representing inductive types as collections of lambda-terms in which the lambda terms are not as easily typeable as in the first method. These two methods provide the means to efficiently formalize syntax-based mathematical algorithms in the lambda calculus.

Using the fact that inductive types can be directly represented in the lambda calculus, Torben Æ. Mogensen in [47] represents the inductive type of lambda terms in lambda calculus itself as well as defines an evaluation operator in the lambda calculus. He thus shows that the replete approach to reasoning about syntax, minus the presence of a built-in quotation operator, can be realized in the lambda calculus. (See Henk Barendregt’s survey paper [3] on the impact of the lambda calculus for a nice description of this work.)

## Metaprogramming

*Metaprogramming* is writing computer programs to manipulate and generate computer programs in some programming language  $L$ . Metaprogramming is especially useful when the “metaprograms” can be written in  $L$  itself. This is facilitated by implementing in  $L$  metareasoning techniques for  $L$  that involve the manipulation of program code. See [15] for a survey of how this kind of “reflection” can be done for the major programming paradigms. Several programming language support metaprogramming including Lisp, Agda [49, 50], F# [65], MetaML [61], MetaOCaml [58], reFLect [35], and Template Haskell [59]. These languages represent fragments of computer code as values in an inductive type and include quotation, quasiquotation, and evaluation operations. For example, these operations are called *quote*, *backquote*, and *eval* in the Lisp programming language. Thus metaprogramming languages take, more or less, the replete approach to reasoning about the syntax of programs. The metaprogramming language Archon [60] developed by Aaron Stump offers an interesting alternate approach in which program code is manipulated directly instead of manipulating representations of computer code.

## Theories of Truth

Truth is a major subject in philosophy [32]. A theory of truth seeks to explain what truth is and how the liar and other related paradoxes can be resolved. A *semantics theory of truth* defines a truth predicate for a formal language, while an *axiomatic theory of truth* [37, 38] specifies a truth predicate for a formal language by means of an axiomatic theory. We have mentioned in Note 12 that an evaluation of the form  $\llbracket \mathbf{A}_\epsilon \rrbracket_o$  is a truth predicate on wffs $_\epsilon$   $\mathbf{A}_\epsilon$  that represent

formulas. Thus  $\mathcal{Q}_0^{\text{uqe}}$  provides a semantic theory of truth via its semantics and an axiomatic theory of truth via its proof system  $\mathcal{P}^{\text{uqe}}$ .

Since our goal is not to explicate the nature of truth, it is not surprising that the semantic and axiomatic theories of truth provided by  $\mathcal{Q}_0^{\text{uqe}}$  are not very innovative. Theories of truth — starting with Tarski’s work [62, 63, 64] in the 1930s — have traditionally been restricted to the truth of sentences, i.e., formulas with no free variables. However, the  $\mathcal{Q}_0^{\text{uqe}}$  semantic and axiomatic theories of truth admit formulas with free variables.

### 13.4 Future Work

Our future research will seek to answer the following questions:

1. Can nontrivial syntax-based mathematical algorithms — such as those that compute derivatives symbolically — be formalized in  $\mathcal{Q}_0^{\text{uqe}}$  in the sense given in [25]?
2. Can a logic equipped with the machinery of  $\mathcal{Q}_0^{\text{uqe}}$  for reasoning about undefinedness and syntax be effectively implemented as a software system?
3. Can the replete approach to reasoning about syntax serve as a basis to integrate axiomatic and algorithmic mathematics?

We will discuss each of these research questions in turn.

#### Formalizing Syntax-Based Mathematical Algorithms

We conjecture that it is possible to formalize nontrivial syntax-based mathematical algorithms in  $\mathcal{Q}_0^{\text{uqe}}$  in the sense given in [25]. We intend to work out the details for the well-known algorithm for the symbolic differentiation of polynomials as described in [25]. First, we will define a theory  $R$  of the real numbers in  $\mathcal{Q}_0^{\text{uqe}}$ . Second, we will define in  $R$  the basic ideas of calculus including the notions of a derivative and a polynomial. Third, we will define a constant in  $R$  that represents the symbolic differentiation algorithm for polynomials. Fourth, we will specify in  $R$  the intended computational behavior of the algorithm and prove that the constant satisfies that specification. Fifth, we will specify in  $R$  the intended mathematical meaning of the algorithm and prove that the constant satisfies that specification. And, finally, we will show how the constant can be used to compute derivatives of polynomial functions in  $R$ .

Polynomial functions are total (i.e., they are defined at all points on the real line) and their derivatives are also total. Hence no issues of definedness arise in the specification of the mathematical meaning of the differentiation algorithm for polynomials. However, functions more general than polynomial functions as well as their derivatives may be undefined at some points. This means that specifying the mathematical meaning of a symbolic differentiation algorithm for more general functions will require using the undefinedness facility of  $\mathcal{Q}_0^{\text{uqe}}$ .



## Implementation of the $\mathcal{Q}_0^{\text{uqe}}$ Machinery

It remains an open question whether a logic like  $\mathcal{Q}_0^{\text{uqe}}$  can be effectively implemented as a computer program. The undefinedness component of  $\mathcal{Q}_0^{\text{uqe}}$  has been implemented in the IMPS theorem proving system [26, 27] which has been successfully used to prove hundreds of theorems in traditional mathematics, especially in mathematical analysis. However, quotation and evaluation would add another level of complexity to a theorem proving system like IMPS that can deal directly with undefinedness.

There are three approaches for implementing the syntax reasoning machinery of  $\mathcal{Q}_0^{\text{uqe}}$ . The first is to implement part of the machinery of  $\mathcal{Q}_0^{\text{uqe}}$  in an existing implementation of Church’s type theory such as John Harrison’s HOL Light [40] in order to conduct experiments concerning reasoning about syntax. For example, a worthy experiment would be to formalize a syntax-based mathematical algorithm like the symbolic differentiation algorithm for polynomials mentioned above. The second is to directly implement  $\mathcal{Q}_0^{\text{uqe}}$  — a version of  $\mathcal{Q}_0^{\text{uqe}}$  with perhaps some practical additions — to test the entire design of  $\mathcal{Q}_0^{\text{uqe}}$ . And the third is to implement  $\mathcal{Q}_0^{\text{uqe}}$ ’s syntax reasoning machinery as part of the implementation of a general purpose logic for mechanized mathematics. We have engineered Chiron [21, 22] to be just such a logic. It contains essentially the same syntax reasoning machinery as  $\mathcal{Q}_0^{\text{uqe}}$ , and we have a rudimentary implementation of it [11].

Implementing the ideas in  $\mathcal{Q}_0^{\text{uqe}}$  will be challenging. Reasoning about the interplay of syntax and semantics in a logic instead of a metalogic is tricky. It is easy for both developers and users to become confused — just ask any Lisp programmer. A practical proof system will require new axioms and rules of inference as well as an effective means to perform substitution in the presence of evaluations. The latter, as we have seen, is fraught with difficulties. Finally, new notation and user-interface techniques are needed to shield the user, as much as possible, from low-level syntactic manipulations.

## Integration of Axiomatic and Algorithmic Mathematics

The MathScheme project [11], led by Jacques Carette and the author, is a long-term project being pursued at McMaster University with the aim of producing a framework in which formal deduction and symbolic computation are tightly integrated. A key part of the framework is the notion of a *biform theory* [10, 20] that is a combination of an axiomatic theory and an algorithm theory. A biform theory is a basic unit of mathematical knowledge that consists of a set of *concepts* that denote mathematical values, *transformers* that denote syntax-based algorithms, and *facts* about the concepts and transformers. Since transformers manipulate the syntax of expressions, biform theories are difficult to formalize in a traditional logic. One of the main goals of the MathScheme is to see if a logic like  $\mathcal{Q}_0^{\text{uqe}}$  that implements the replete approach to syntax reasoning can be used to develop a library of biform theories.

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