Teaching and Learning Models for Mathematics using Mathematica

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Introduction

When one studies a mathematical problem nowadays, tools such as calculators and computers are available for students and teachers. Students can be actively engaged in reasoning, communicating, problem solving, and making connections with Mathematics and other disciplines [3]. Learning mathematical properties and principles can be enhanced through visualization using computer graphics. Complicated calculations that are time-consuming by paper and pencil can be carried out instantaneously using Mathematica or some other mathematical software package. Some writers have asserted that the computational medium is affecting not only the means by which Mathematics is transmitted and learned, but also the content and context in which it is important [2].

The eminent mathematician and educator Gail S. Young [13] has compared the advent of the computer to the introduction of Arabic numerals into Europe or the invention of the calculus in its impact. He points out similarities with those earlier "revolutions" by means of which "hard problems became easy and solvable not only by an intellectual elite but by a multitude of people without special mathematical talent; problems arose that had not been previously visualized, and their solutions changed the entire level of the field". In the National Research Council (USA) report *Everybody Counts* [9] we also read " … computers have posed new problems for research [in Mathematics], supplied new tools to solve old problems, and introduced new research strategies".

For some years now people have been developing software for use in the mathematics classroom. Packages have varied greatly in their ease of use, their motivational appeal, their level of intellectual demand and their cross-platform accessibility [7]. *Everybody Counts*, taking a problem-solving view of mathematics, tells us that "research on learning shows that most students cannot learn mathematics effectively by only listening and imitating" [9]. Technology brings to students and teachers the opportunity to individualize learning -- to generate illustrative examples, to follow interesting topics to the desired depth, to choose their own problems and then gather the tools for solving them.

Information technologies have transformed the workplace, but not yet the schools. The various teaching-learning models which involve mathematical software need to be developed further.

In this paper, we give examples of models we have created for use in university mathematics courses. We explain the concept of a linear transformation, investigate the roles of each component of 2x2 and 3x3 transformation matrices, consider the relation between sound and trigonometry, visualize the Riemann sum, the volume of surfaces of revolution and the area of unit circle, estimate the area of a figure by probabilistic simulation and illustrate various examples of parametric cycloids. We give interesting, visual, meaningful and effective models for teaching the above subjects, which are obtained by the powerful functions of Mathematica such as animation of graphics, variety of visualization and speed of computation. This paper illustrates how one can use Mathematica to visualize abstract mathematical concepts, thus enabling students' effective understanding in the mathematics classroom. Development of these kinds of teaching and learning models can stimulate the students' curiosity about Mathematics and increase their interest.

The Mathematica code described in this paper is available from the authors. However, for the sake of brevity and readability, most of the code has been omitted from the paper.

Model 1: Linear transformations

1. Plane linear transformations

A plane transformation T: $E^2 \to E^2$ is represented by

$$(1.1) \qquad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

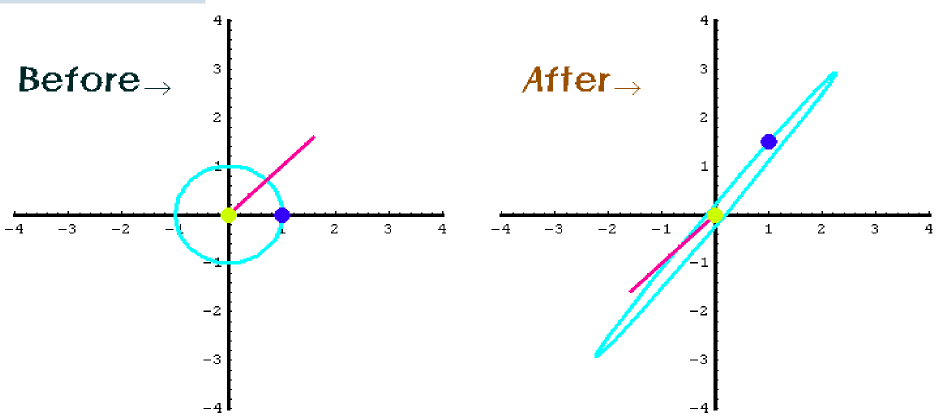
and the coefficient matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is said to be the *representation matrix* of T. In order to investigate the properties of T and the roles of each component of T, we define the Module function trans[x_, y_, z_], where x is a 2 × 2 matrix and y, z are curves. We shall study how the curves are transformed by the transformation matrix x. Using this tool, we can to visualize various aspects of linear transformations: the identity transformation, reflections, rotations, homotheties, and inverse transformations.

We take as our curves, the unit circle centred at the origin and the line y=x.

$$x = \begin{pmatrix} 1 & -2 \\ 1.5 & -2.5 \end{pmatrix};\ y = \begin{pmatrix} \text{Cos}[t] \\ \text{Sin}[t] \end{pmatrix};\ z = \begin{pmatrix} t \\ t \end{pmatrix};$$

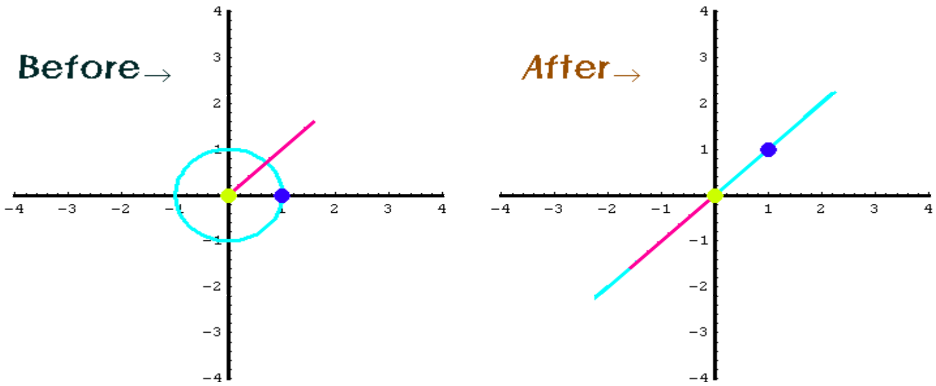
We compare the two graphs before transformation with those after transformation.

```
trans[x, y, z]
```



With this, and other choices of z, we illustrate the fact that nonsingular linear transformations take circles to ellipses and lines to lines. If T is singular (but nonzero), we note that the image of T is 1-dimensional. The circle is collapsed to a line segment.

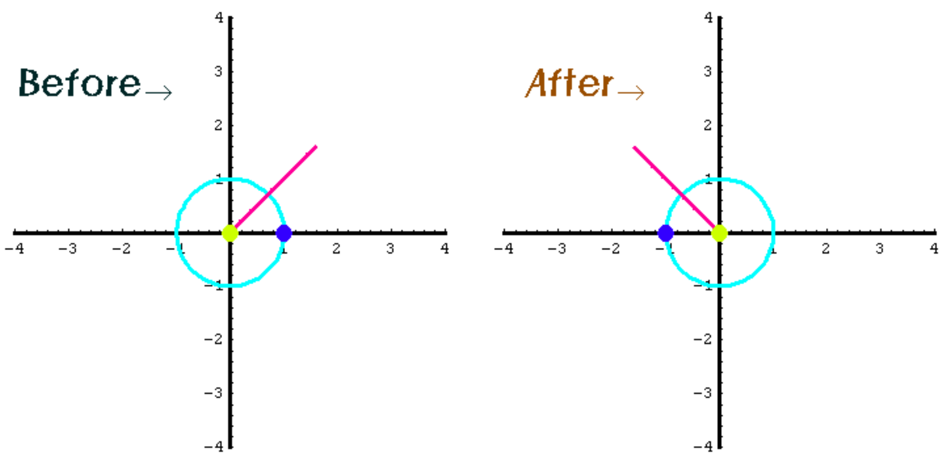
$\texttt{x1} = \begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix}$; trans[x1, y, z]



We consider the identity transformation, a homothety (central stretching), a reflection and a rotation.

trans[$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, y, z]   trans[$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$, y, z]

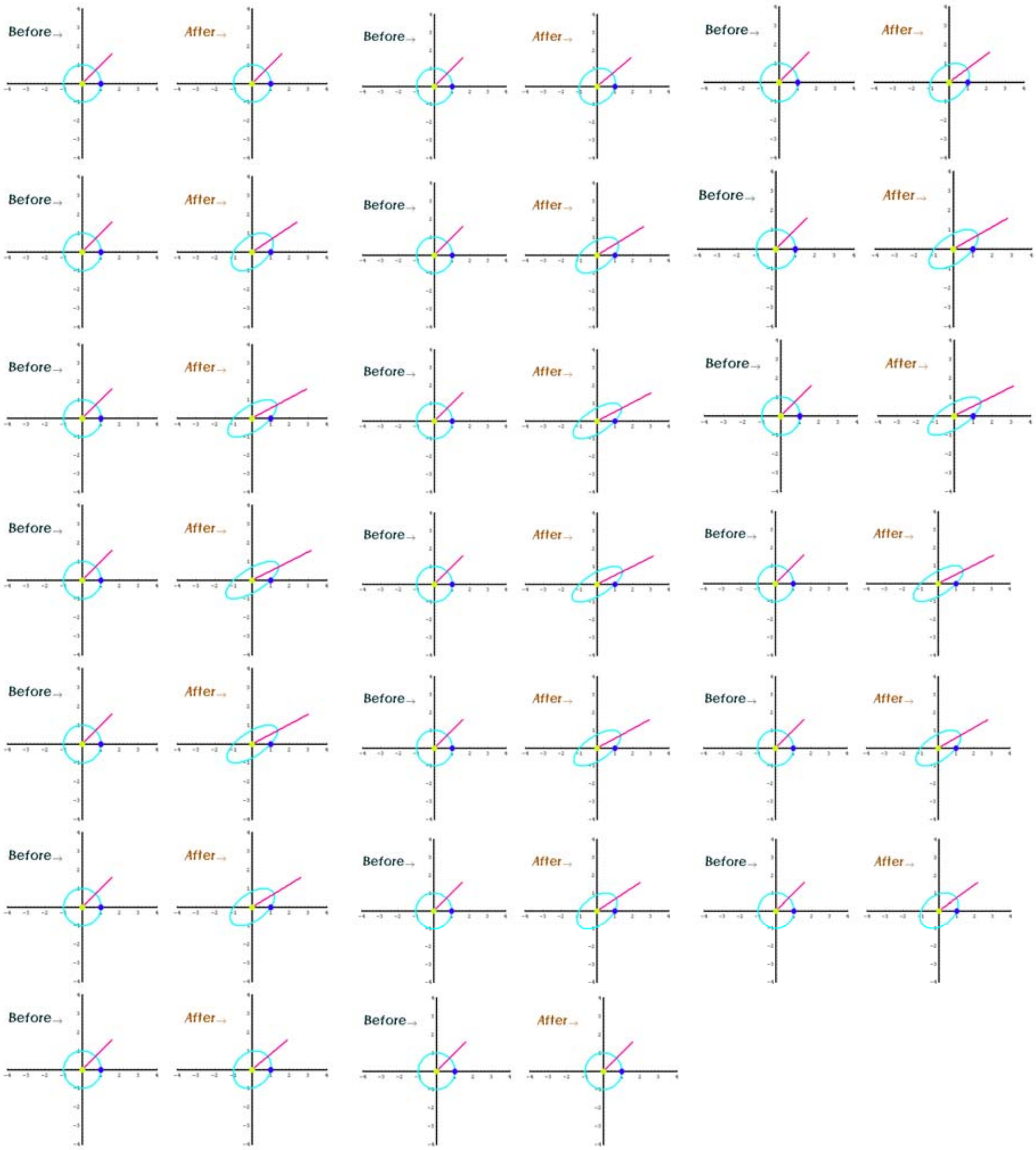
trans[$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, y, z]



trans[$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, y, z]   trans[$\begin{pmatrix} \text{Cos}[45\,°] & -\text{Sin}[45\,°] \\ \text{Sin}[45\,°] & \text{Cos}[45\,°] \end{pmatrix}$, y, z]

What can be said about the role of each (i, j) -component of the transformation matrix? We can animate how the graphs of the two curves when the (1,2) and (2,1)-components (in various combinations) range from -1 to 1. To do this we use the Table function in Mathematica.

Table[trans[$\begin{pmatrix} 1 & -w^2 + 2\,w \\ 0 & 1 \end{pmatrix}$, y, z], {w, 0, 2, 0.1}];



Table[**trans**[$\begin{pmatrix} 1 & w^2 - 2w \\ 0 & 1 \end{pmatrix}$, **y**, **z**], {**w**, **0**, **2**, **0.1**}];

Table[**trans**[$\begin{pmatrix} 1 & 0 \\ -w^2 + 2w & 1 \end{pmatrix}$, **y**, **z**], {**w**, **0**, **2**, **0.1**}];

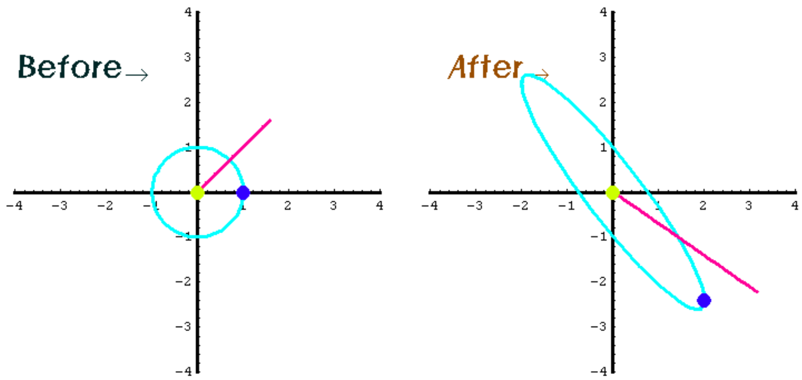
Table[**trans**[$\begin{pmatrix} 1 & 0 \\ w^2 - 2w & 1 \end{pmatrix}$, **y**, **z**], {**w**, **0**, **2**, **0.1**}];

Table[**trans**[$\begin{pmatrix} 1 & -w^2 + 2w \\ -w^2 + 2w & 1 \end{pmatrix}$, **y**, **z**], {**w**, **0**, **2**, **0.1**}];

Table[**trans**[$\begin{pmatrix} 1 & w^2 - 2w \\ w^2 - 2w & 1 \end{pmatrix}$, **y**, **z**], {**w**, **0**, **2**, **0.1**}];

The function *trans* can also be used to illustrate the fact that the composition of linear transformations is not commutative.

$$\text{trans}\left[\begin{pmatrix} -1 & 0 \\ 1.2 & 1 \end{pmatrix}\cdot\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}, y, z\right]$$



$$\text{trans}\left[\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} -1 & 0 \\ 1.2 & 1 \end{pmatrix}, y, z\right]$$

After this experiment, many students will have a vivid reminder of the fact that matrix multiplication and the composition of functions are not commutative.

Finally, we replace the line by the parabola $y=x^2$ to illustrate the behaviour of transformations in a slightly more complicated situation. Experimenting with this and other examples, students may strengthen their understanding of the plane transformation geometry.

2. Space linear transformation

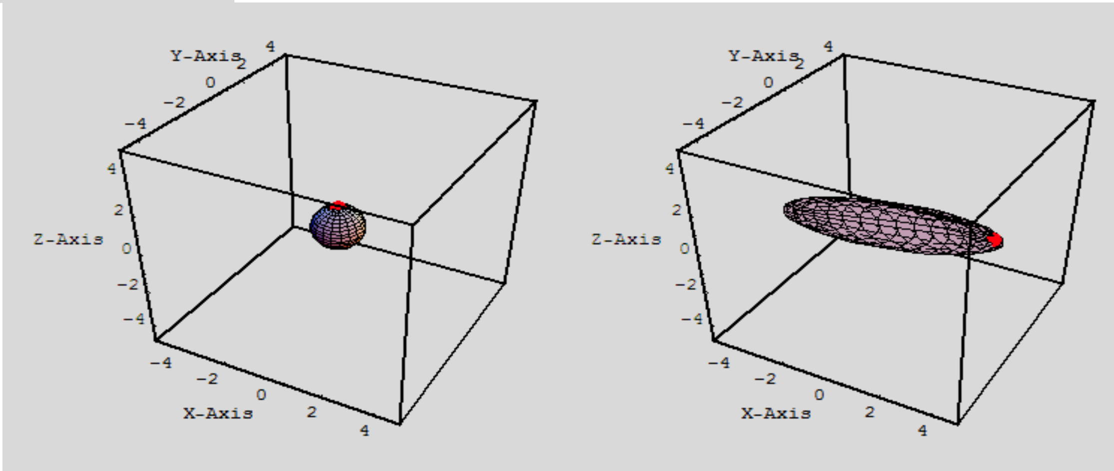
A 3×3 matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \textit{for any real } a_{ij}\,(1\le i,j\le 3). \tag{1.2}$$

represents a space linear transformation T: $E^3 \to E^3$. We define the function trans3d[x_, y_], where x is a 3×3 matrix and y is a surface. Using the same process as for a plane linear transformation, we can study various aspects of a space linear transformation.

```
Clear[y]; x2 = {{1, -1, 3}, {0, -1, 2}, {-1, 0, -1}}; y = {{Cos[t] Cos[u]}, {Sin[t] Cos[u]}, {Sin[u]}};
trans3d[x2, y]
```

$$x2 = \begin{pmatrix} 1 & -1 & 3 \\ 0 & -1 & 2 \\ -1 & 0 & -1 \end{pmatrix}; \quad y = \begin{pmatrix} \mathrm{Cos}[t]\,\mathrm{Cos}[u] \\ \mathrm{Sin}[t]\,\mathrm{Cos}[u] \\ \mathrm{Sin}[u] \end{pmatrix};$$




```
Table[trans3d[{{1, w, 0}, {2 - w, 1, 0}, {0, 1 - 0.5 w, 1}}, y], {w, 0, 2, 0.1}];
```

$$\mathrm{Table}\left[\mathrm{trans3d}\left[\begin{pmatrix} 1 & w & 0 \\ 2-w & 1 & 0 \\ 0 & 1-0.5\,w & 1 \end{pmatrix}, y\right], \{w, 0, 2, 0.1\}\right];$$

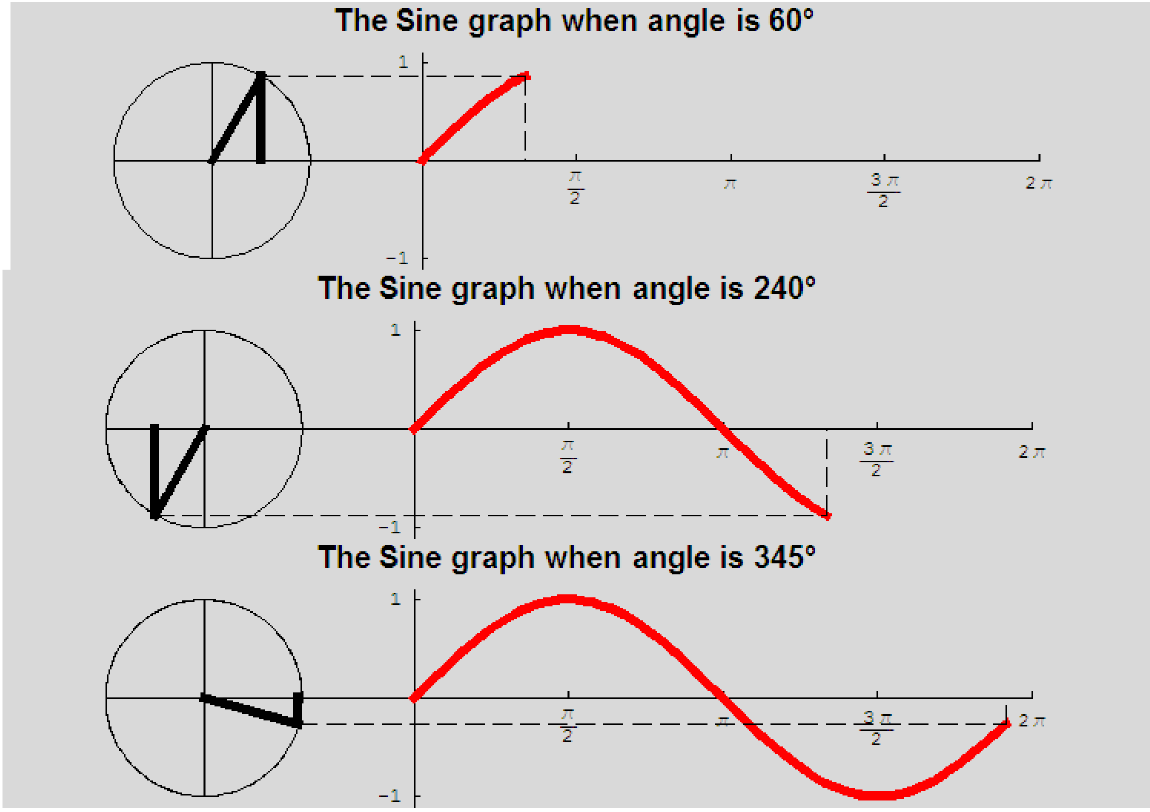
As an example, we show how a sphere is transformed by a one particular transformation and also by a parametrized family of transformations. The sphere is transformed into a family of ellipsoids and as the parameter passes through 1, the ellipsoid is collapsed to a plane ellipse. Presentations of these examples to students will point the way to many further experiments that are likely to be rewarding.

Model 2 : Music

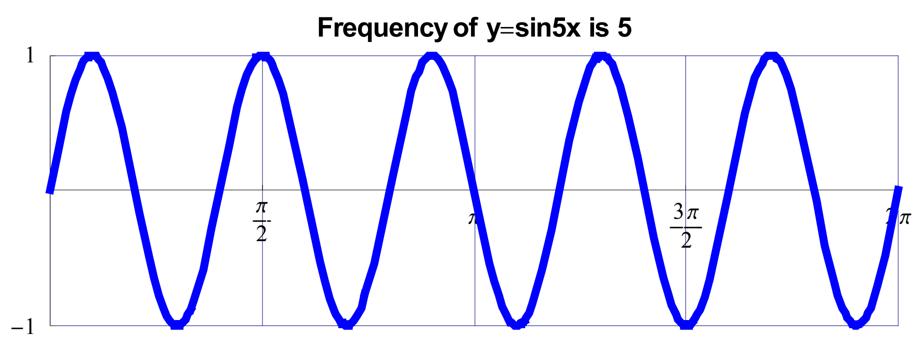
In this section we give various animations of trigonometric functions and show how musical notes can be produced using, for example, the sine and exponential functions.

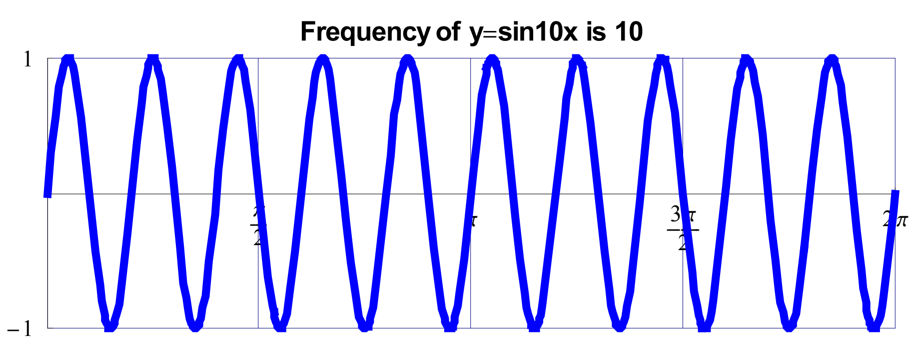
First, we define the function SinPlot which can be used to illustrate the relationship between the sine graph and the unit circle. Then we can animate the drawing of the sine graph as its argument runs from 0 to 2π.

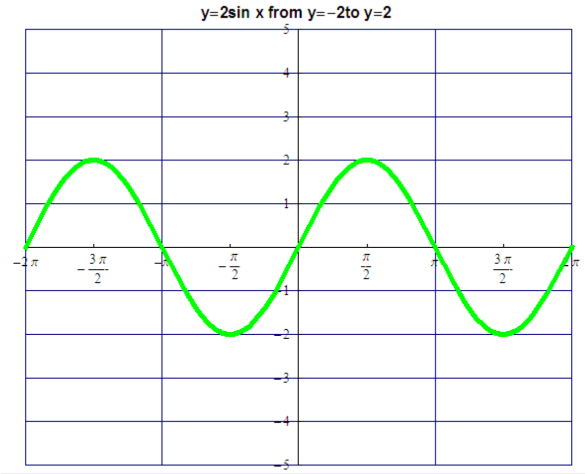
```
Do[SinPlot[θ], {θ, (π / 180) * 15, 2 π, (π / 180) * 15}];
```

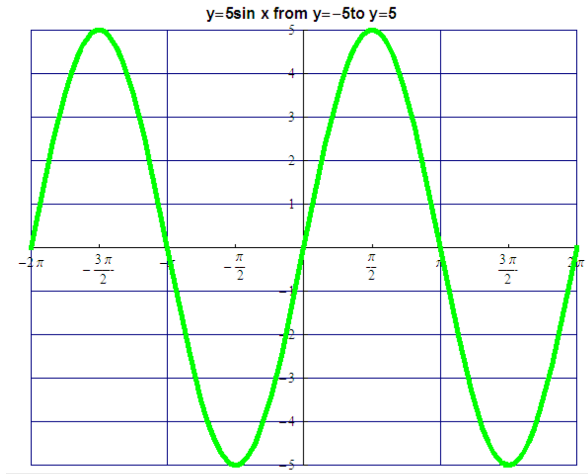


As an aid in explaining the relationship between the sine function and sound production, we plot graphs of the form a sin(x) and sin(b x) for real a, b. This illustrates the effect of frequency and amplitude on the graph.









Mathematica has a Play function which is analogous to Plot but produces sound instead. For example, we can produce a sound by running the following:

$$\text{Play}\left[\frac{20\,\text{Sin}[100 + x^3 + \text{Sin}[10 + x^3] + 20\,\text{Cos}[100\,x^3 + \text{Sin}[10\,x^3]]]}{100 + x\text{^}2 + 100\,\text{Sin}[x\text{^}3]},\ \{x, 0, 5\}\right];$$

```
Play[Sin[5000 x], {x, 0, 3}];
```

In order to represent musical notes, we take account of the familiar 12 tone scale and use a frequency that increases by the same multiplicative factor at each step. The factor is chosen so to be 2^(1/12) so that when the octave is reached (12 steps) the frequency will have been doubled. The following is an example that produces 6 notes.

$$
\begin{aligned}
\text{Play}\Big[\text{Which}\Big[&0 < x < 1,\ \frac{\text{Sin}\left[1000\,x\,2^{\frac{-4}{12}}\right]}{e^{x\,2^{\frac{-4}{12}}}}\ \text{Sin}\left[1000\,x\,2^{\frac{-4}{12}}\right],\\
&1 < x < 1.5,\ \frac{\text{Sin}\left[1000\,(x-1)\,2^{\frac{3}{12}}\right]}{e^{(x-1)\,2^{\frac{3}{12}}}}\ \text{Sin}\left[1000\,(x-1)\,2^{\frac{3}{12}}\right],\\
&1.5 < x < 2,\ \frac{\text{Sin}\left[1000\,(x-1.5)\,2^{\frac{0}{12}}\right]}{e^{(x-1.5)\,2^{\frac{0}{12}}}}\ \text{Sin}\left[1000\,(x-1.5)\,2^{\frac{0}{12}}\right],\\
&2 < x < 4,\ \frac{\text{Sin}\left[1000\,(x-2)\,2^{\frac{3}{12}}\right]}{e^{(x-2)\,2^{\frac{3}{12}}}}\ \text{Sin}\left[1000\,(x-2)\,2^{\frac{3}{12}}\right],\\
&4 < x < 5,\ \frac{\text{Sin}\left[1000\,(x-4)\,2^{\frac{8}{12}}\right]}{e^{(x-4)\,2^{\frac{8}{12}}}}\ \text{Sin}\left[1000\,(x-4)\,2^{\frac{8}{12}}\right]\Big],\\
\{x, 0, 5\}\Big]
\end{aligned}
$$

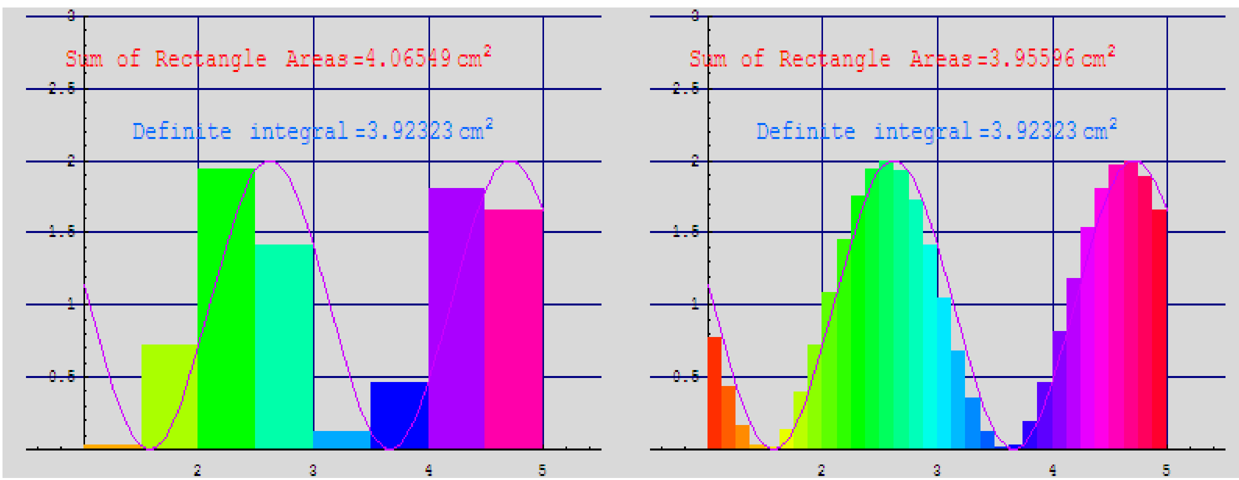
Using the same method, we have prepared a function that plays the song "songaji" which is popular in Korea.

Model 3 : Integral Calculus

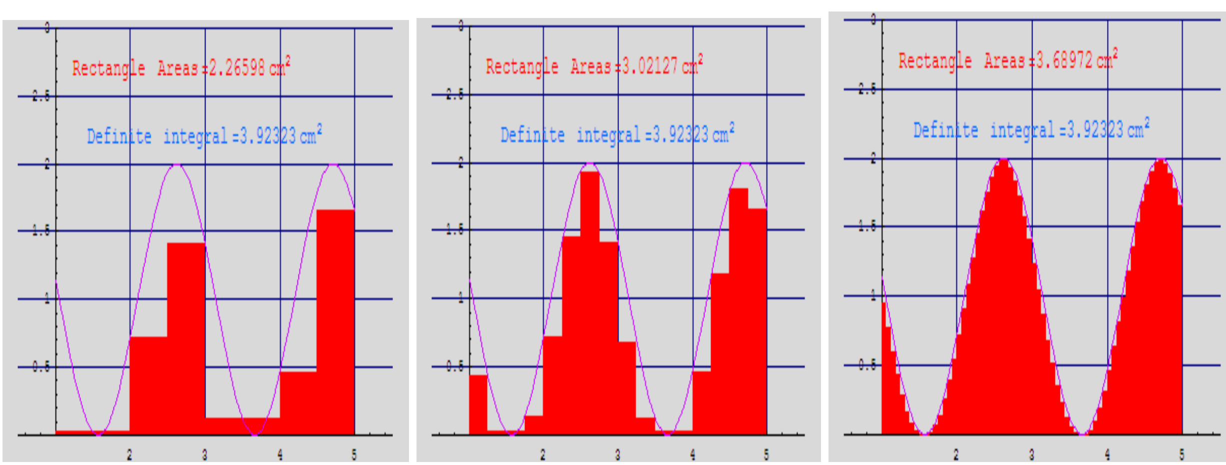
1. Riemann sums

A common use/demonstration of integration is to determine the area under a curve and the volume under a surface. In this section we introduce methods for the visualization of these two concepts. For a given function and interval, we plot the graph of the function and calculate numerically the area under it. First, we consider the area under the curve f(x) = 1 + sin 3x.

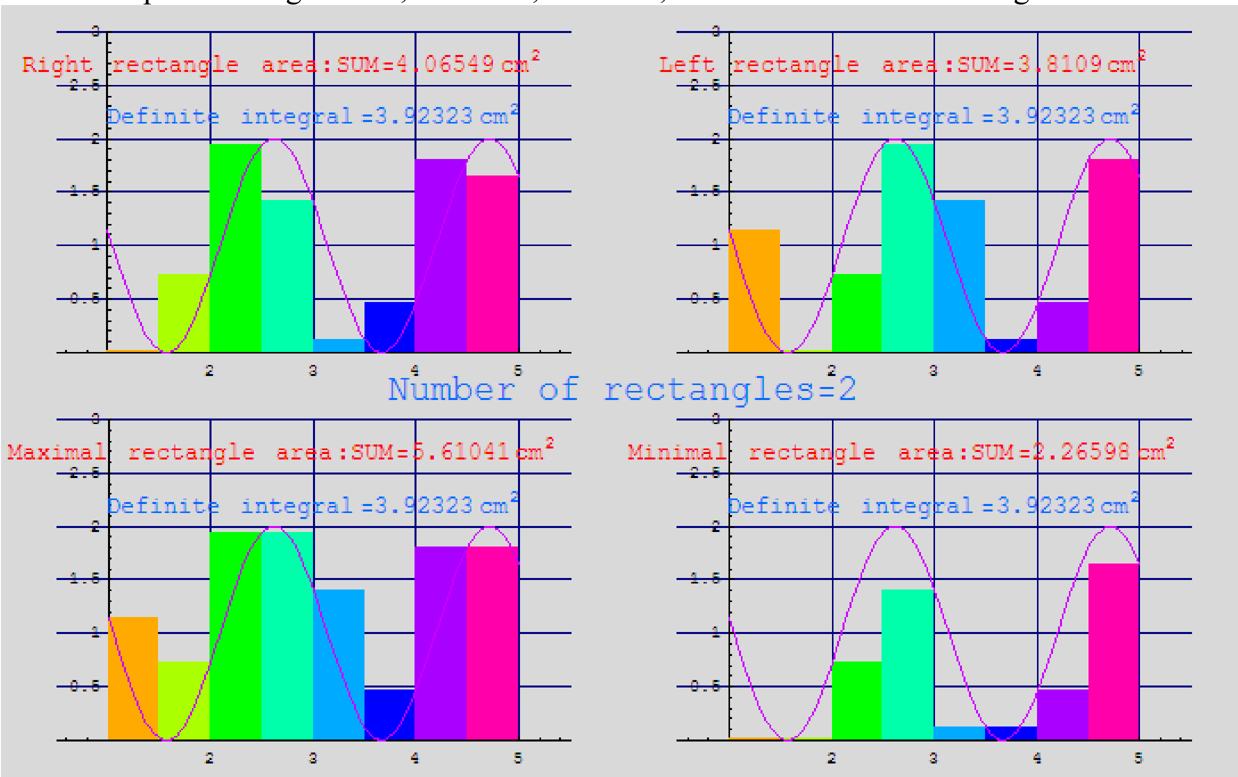
A natural way of estimating this area is to first subdivide the domain into evenly spaced intervals and then add together the areas of certain approximating rectangles. Now, in order to see that how the sum of the rectangle areas approximates the definite integral, we can compare right-hand, left-hand, maximal and minimal Riemann sums for many different numbers of subdivisions. This displays the Riemann sums for f(x) using the right-hand method with 8 and 32 subdivisions.



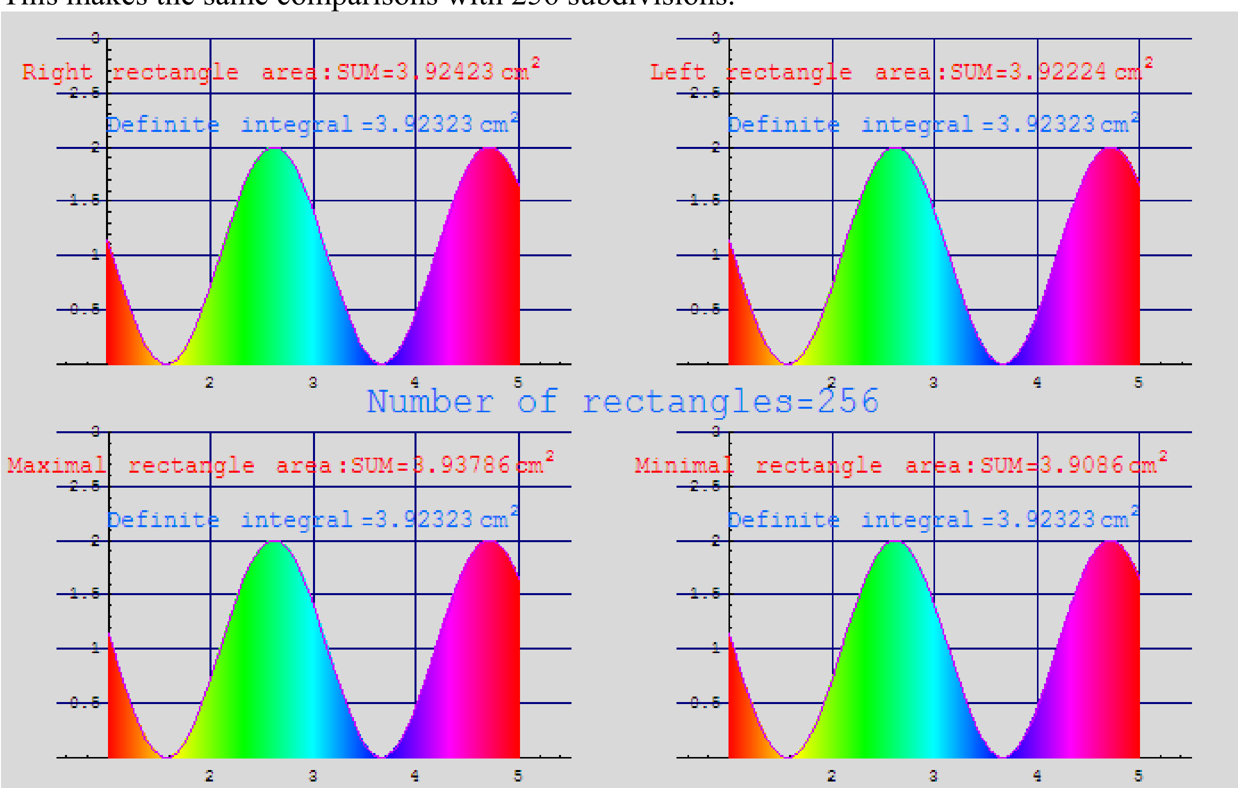
This displays the minimum method with 8, 16, and 64 subdivisions.



This compares the right-hand, left-hand, maximal, and minimal methods using 8 subdivisions.



This makes the same comparisons with 256 subdivisions.



Moreover, using the function Block we can allow the user to specify any function, any interval and any number of subdivisions, and then produce the various Riemann sums.

2. Volume of a surface of revolution

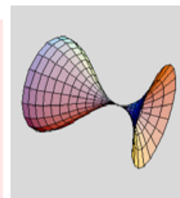
Since the volume of a surface of revolution is the integral of an area of cross sections orthogonal to the axis of revolution, and the cross section is a circle, it is relatively easy to visualize it. As an example, we give a method to visualize the volume of a wine glass.

First, we choose a suitable curve .

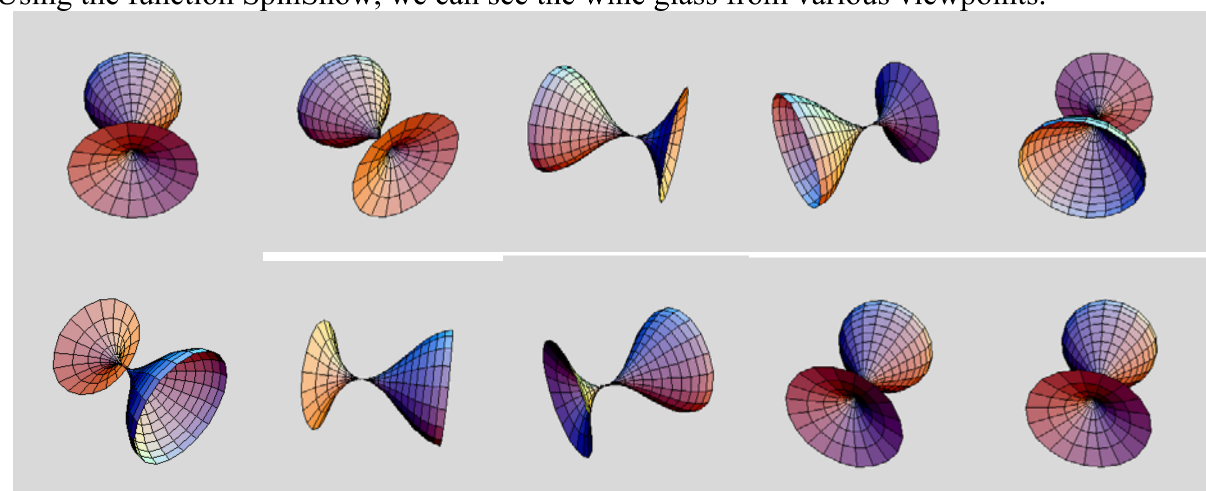
```
Plot[1/3 (x^3 - 6 x^2 + 9 x), {x, 1, 4}, PlotStyle → {Hue[0.6], Thickness[0.012]}];
```

By revolving it around the x-axis, we obtain a wine glass.

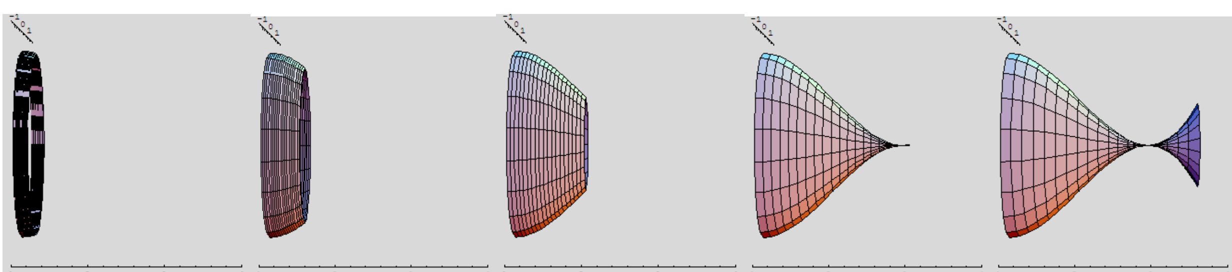
```
g = ParametricPlot3D[{t, 1/3 (t^3 - 6 t^2 + 9 t) Cos[u], 1/3 (t^3 - 6 t^2 + 9 t) Sin[u]}
        , {t, 1, 4}, {u, 0, 2 π}, Boxed → False
        , Axes → False, PlotPoints → Automatic
        , PlotRange → {{1, 4}, {-1.5, 1.5}, {-1.7, 1.7}}];
```

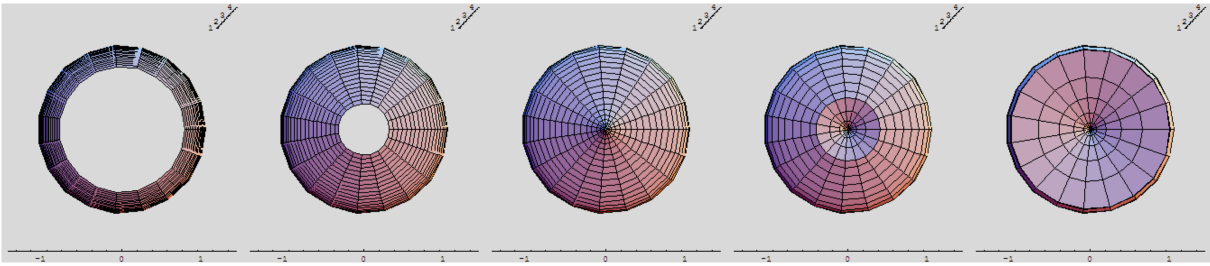


Using the function SpinShow, we can see the wine glass from various viewpoints.

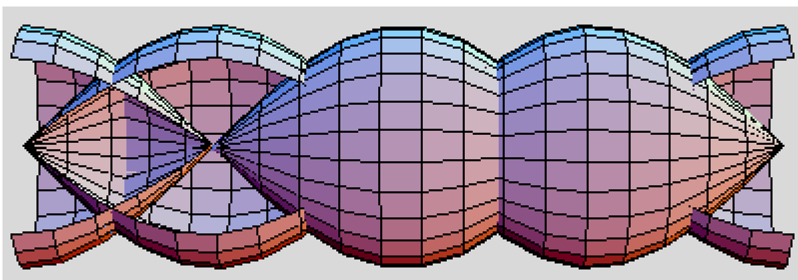


The following animations illustrate computation of the volume of the surface of revolution the integration of the area of the circular cross-section.

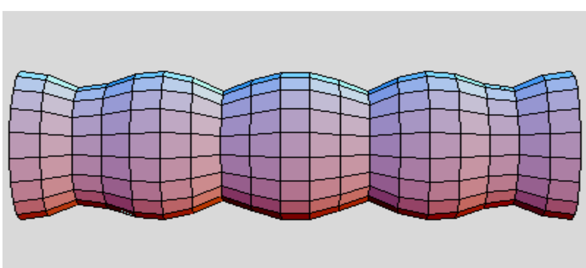
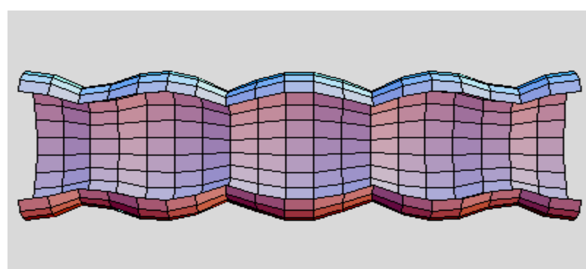




On the other hand, we can attempt to find another example to visualize the surface of revolution obtained by revolving two curves sin x and cos x around the x-axis. Of course, the two curves intersect transversely so that the two surfaces interpenetrate. The following "cut-away" view illustrates the "inner" and "outer" surfaces.



This demonstration leads the student to two observations: (i) it is sufficient and simpler to use the absolute value of the function being revolved (here use |sin x| and |cos x|); (ii) in order to get a single surface, we can choose for each x, the maximum of |sin x| and |cos x| as our function value. This has the effect of eliminating the inner surface. This is shown below, both with an outer view and a cut-away view.

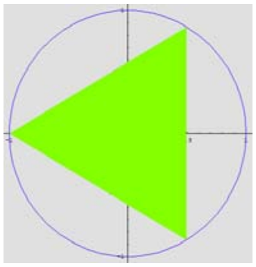
 

Model 4 : Computing the area of the unit circle

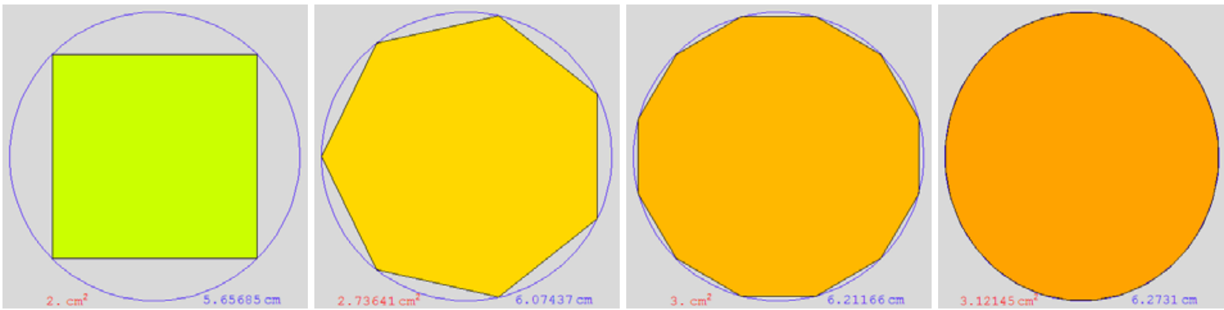
In this section, we show how Mathematica can be used to illustrate several different computations of the area of the unit circle.

1. Using inscribed regular polygon

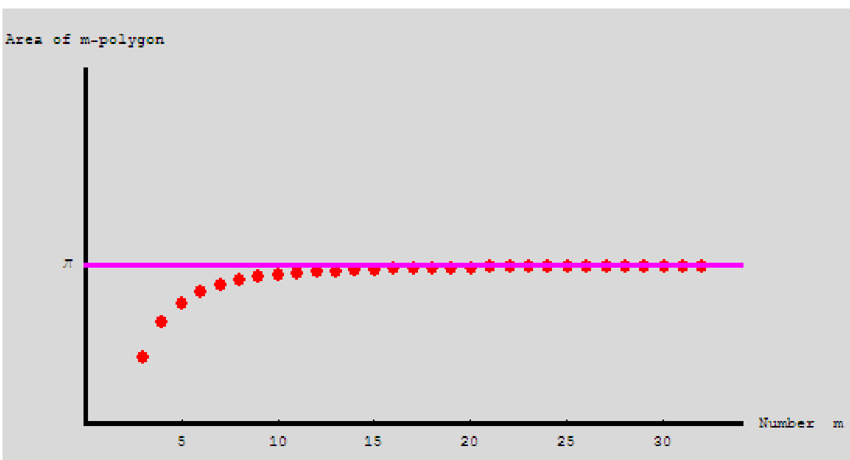
One familiar way to define the area of a circle is as the limit of areas of inscribed regular polygons (m-gons) as the number of sides m tends to infinity. We begin with m=3.



We now consider the inscribed m-gon (m = 3, 4, ... , 32) of the unit circle.



The following show that the area and the perimeter of an m-gon approach the area of the unit circle (π) and the perimeter of unit circle (2π) as m increases.



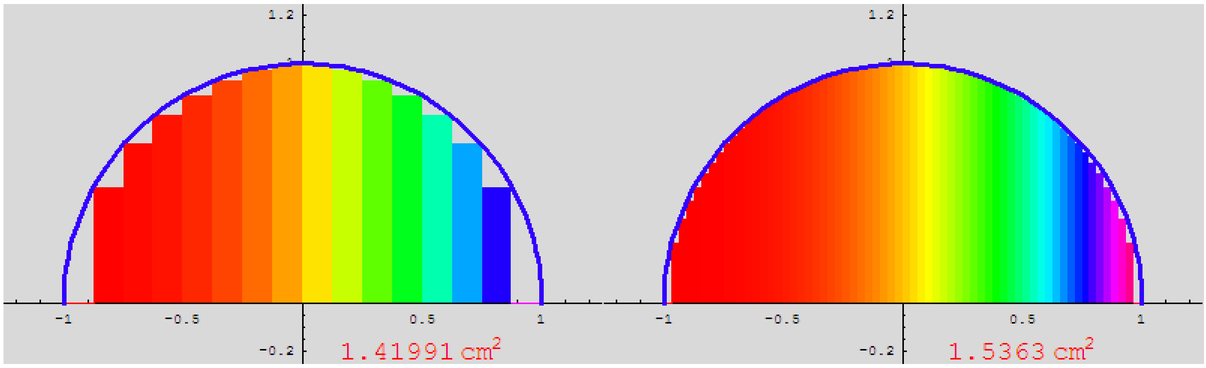
2. Using Riemann sums

A second method of computing the area is to treat the semicircle as the graph of a function. We begin by drawing a semi-unit circle.

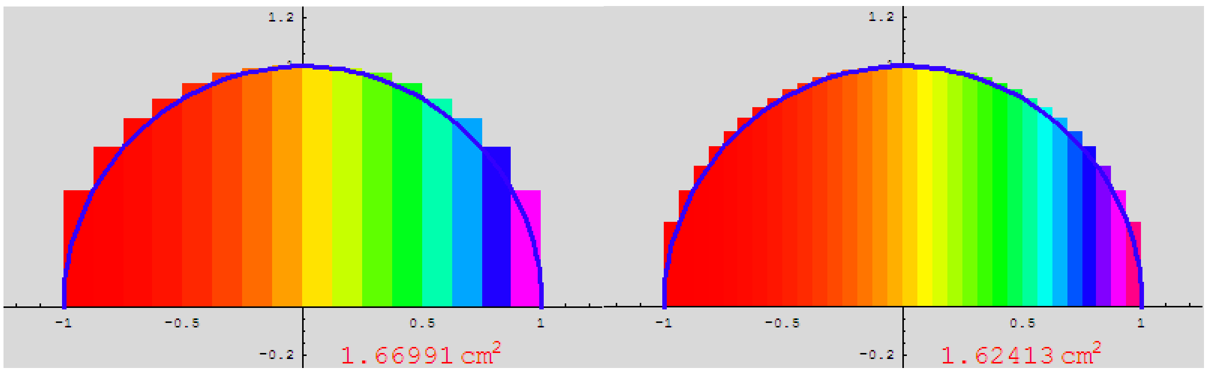
$f[x\_] := \sqrt{1 - x^2}$

```
cir3 = Plot[f[x], {x, -1, 1}, AspectRatio → Automatic,  PlotRange → {{-1.25, 1.25}, {-0.25, 1.25}},
           ImageSize → 400, Axes → True,  PlotStyle → {Thickness[0.01], Hue[0.7]}];
```

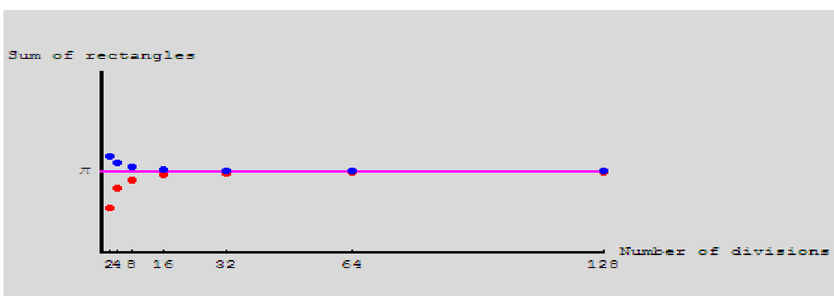
First, consider the total area of the inscribed rectangles in the unit semi-circle using 2^m subdivisions.



Next, consider the sum of outer rectangles covering the unit semi-circle with the same number of subdivisions.



The following illustrates that the total areas of inscribed rectangles approaches the area of unit semi-circle ($\pi/2$) as m increases.



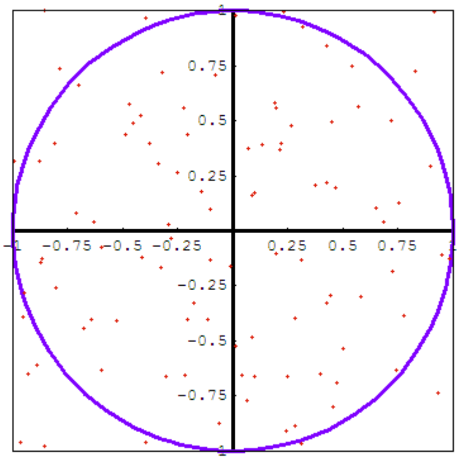
Model 5: Estimating the area of a figure by probabilistic simulation [1]

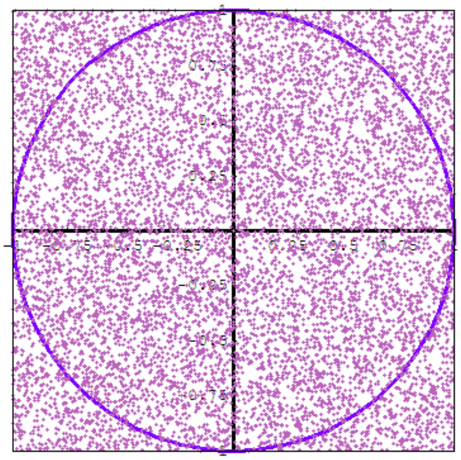
1. Estimating the area of the unit circle

We draw a square of side 2 and the inscribed unit circle of the square. Then we design Mathematica programming which scatters points randomly (uniformly distributed) in the square and count the number of points within the circle. We can estimate the area of unit circle by computing (the area of the square)×(numbers of points in unit circle)÷(numbers of points in square).

(1) Estimating the area by a single sample of various sizes.

. We try 5 different cases using 500, 1000, 2000, 10000, and 100000 sample points respectively. Note that in these displays “data numbers” means sample size and “repeating numbers” means number of repetitions (samples).





Data numbers?=10000

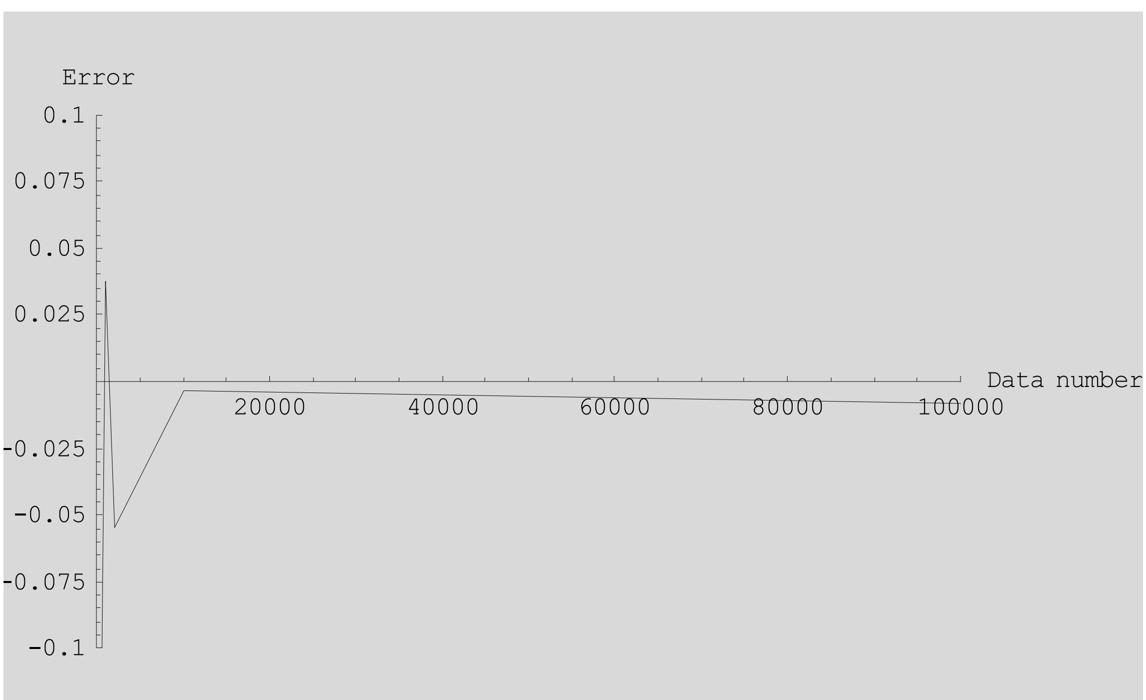
Numbers inside circle?=7826

Estimation of area =3.1304000000000000000

Since the error is $\pi - \frac{3913}{5000} \times 4$

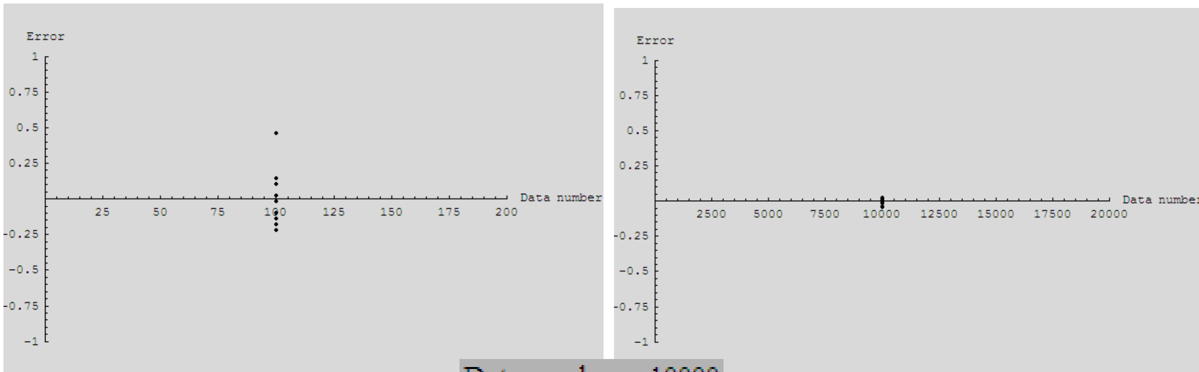
The error of area=0.0111926535897932384 63

The following graph shows the results. Note that the error does not appear to decrease with sample size.



(2) Estimating the area by repeated samples

Next, we choose 10 samples and record the average of the results. We use the same sequence of sample sizes. Then we can see that the error decreases as the sample size increases.

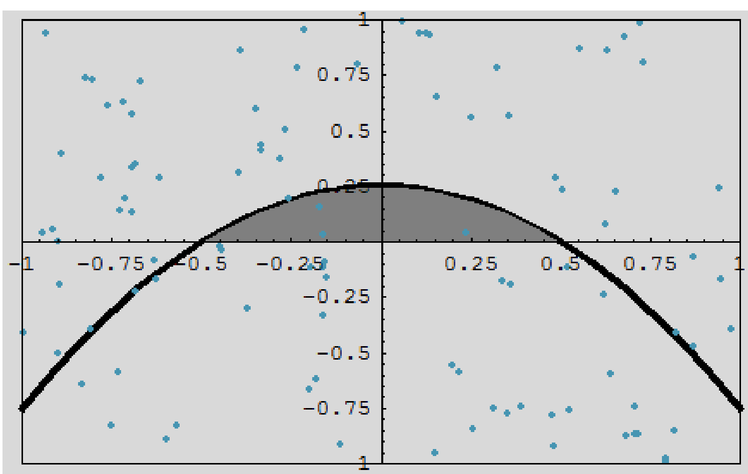


Data numbers=10000
Repeating numbers=10
10, Mean of unit circle area=3.14872
Dispersion of area=0.000455602
Declination of area=0.0213448

2. Estimating the area under a graph using probabilistic simulation.

We consider a function $f(x)=-x^2+1/4$ and estimate the area of the figure enclosed by f(x) and x-axis.

Again, we illustrate the effect of repeated sampling and of sample size.

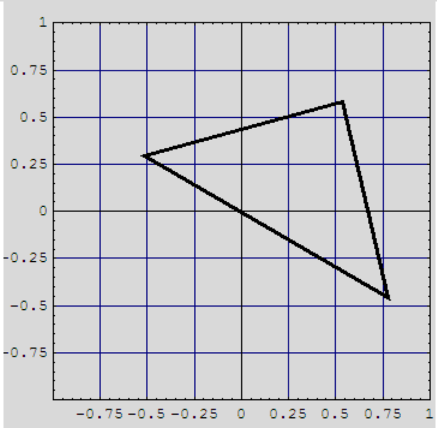


3. Computing the area of a triangle

In this section, show how the user can be allowed to specify a triangle, compute the area using the familiar formula, and then estimate it using probabilistic simulation.

First, a coordinate grid is presented and we choose three points by clicking. The coordinates must then be copied to the list data1. Mathematica then draws the triangle.

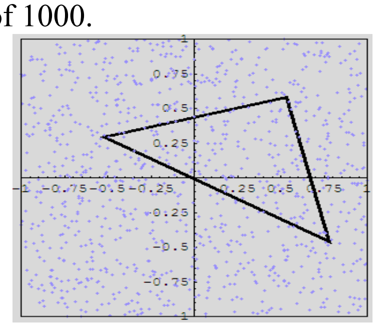
```
data1 = {{0.534383, 0.580189}, {-0.517162, 0.293404}, {0.776856, -0.458411}}
data = Append[data1, {Part[data1, 1, 1], Part[data1, 1, 2]}];
ListPlot[data, PlotJoined → True, PlotRange → {{-1, 1}, {-1, 1}}, Frame → True,
  GridLines → Automatic, AspectRatio → 1, PlotStyle → {Thickness[0.01]}];
```

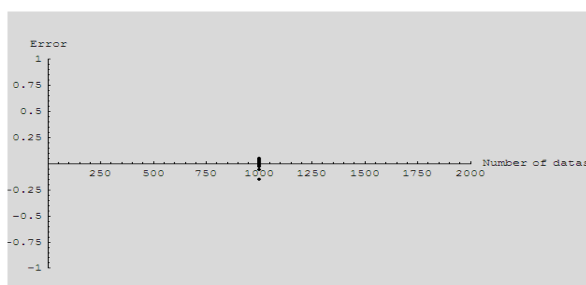


We calculate the area of the triangle.

```
GetArea[pt_] := Module[{x, y}, x = Transpose[pt][[1]]; y = Transpose[pt][[2]];
   n = Length[data];
   Return[
    1 / 2
     Sum[i=1 to n-1] (0.5 (x[[i + 1]] + x[[i]]) (y[[i + 1]] - y[[i]]) -
        0.5 (y[[i + 1]] + y[[i]]) (x[[i + 1]] - x[[i]]))]];
getarea = GetArea[data]
```

By the same method as used in the preceding two sections we can estimate the area of the triangle and we compare the results with and without repeating samples. We use a sample size of 1000.





Random number=1000

Number inside triangle=146

Area of triangle=0.580836

Estimation of area=0.58400000000000000000

Error=−0.00316387

Random number?=1000

Repeating number=10

Area of triangle=0.580836

Mean of areas=0.602

Declination of areas=0.00492889

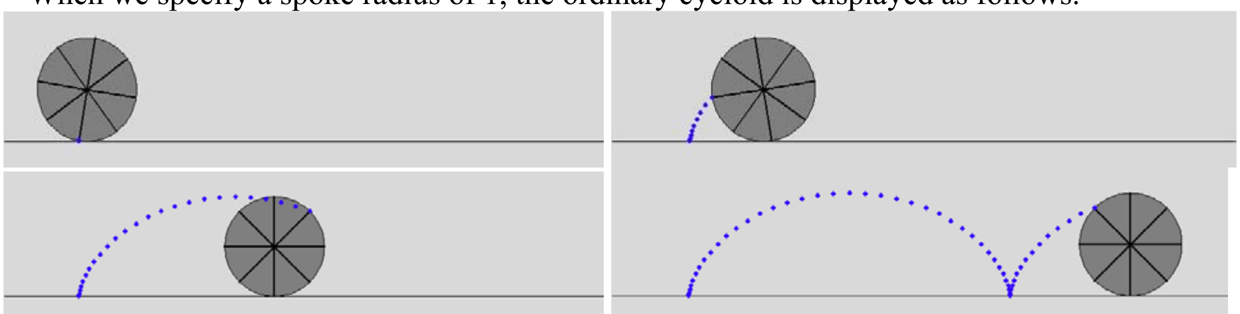
This idea raises the problem of estimating the area of any polygon.

Model 6: Visualization of the cycloid

1. Cycloid

A cycloid is the path traced out by a point on the circumference of a wheel which rolls along a line in the plane. The cycloid and its generalizations provide a nice opportunity for visualization. We start immediately with a generalization in which the point need not be on the circumference but can be any point on a "spoke" of the wheel. The circle is taken to have radius 1 but the user is allowed to specify any radius r for the spoke. The traditional cycloid corresponds to r=1.

When we specify a spoke radius of 1, the ordinary cycloid is displayed as follows.

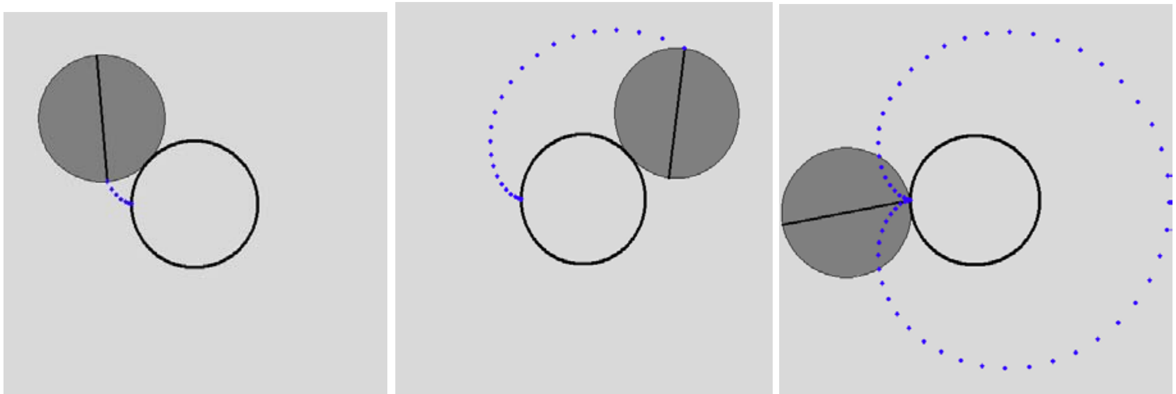


For any value of r, the equations of the curve are as follows:
$x = t - r \sin t$, $y = 1 - r \cos t$. When $r>1$, the curve will have self-intersections (loops).

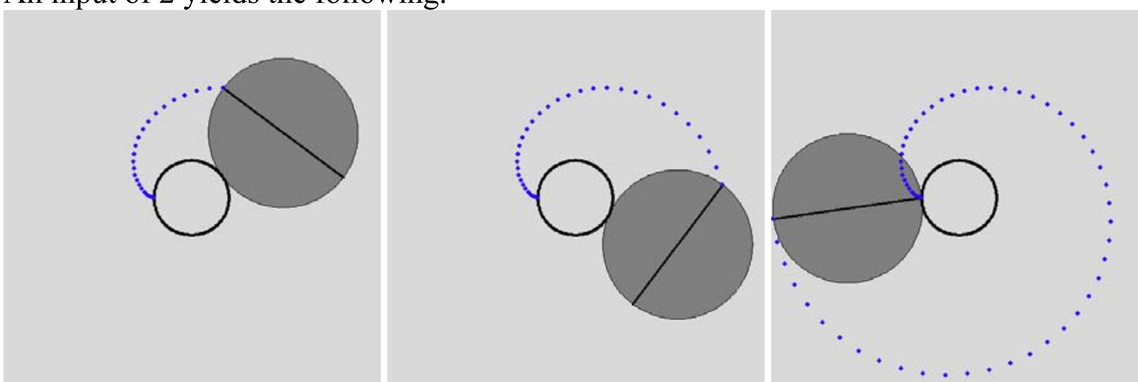
2. Epicycloid and Hypocycloid

A further variation on the cycloid is obtained by allowing the wheel to roll along a circle, either outside (epicycloid) or inside (hypocycloid). Again, we do not insist that the point lie on the circumference (as it would in the classical versions of these curves) but can be any point on a spoke of the wheel. A positive input is interpreted as outside and a negative input is interpreted as inside. More details and equations for these curves may be found in [6].

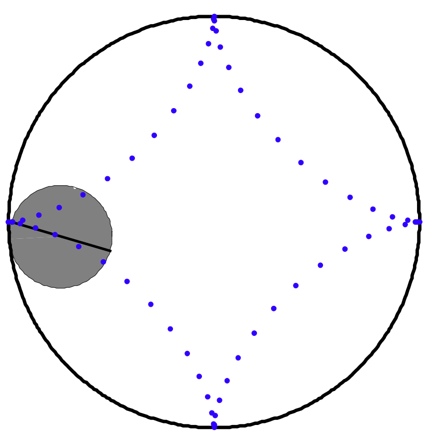
An input of 1 yields the following, which we recognize as a cardioid.



An input of 2 yields the following.



We can also try an input of –0.25, which yields the familiar 4-cusped hypocycloid.



Conclusion

It is clear that computer systems are presenting new challenges and opportunities in the mathematics classroom. Their ubiquity has the potential to affect curriculum and teaching-learning methods both in high schools and universities. As Muller [8] observes, new educational technologies "will erect new barriers for some people, while [freeing others] to explore the world of Mathematics in a very different environment."

In this paper, we have presented six models designed to encourage effective teaching and learning in the mathematics classroom. Since abstract mathematical definitions and concepts need to be represented easily in order to facilitate the student's understanding, mathematical software and other technologies may stimulate better mathematics education. The more novel the environment is, the better the chance to arouse the students' curiosity. It may well be that these experiments raise more questions than they answer.

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