Andrews' Type Theory with Undefinedness[1]

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ABSTRACT. $\mathcal{Q}_0$ is an elegant version of Church's type theory formulated and extensively studied by Peter B. Andrews. Like other traditional logics, $\mathcal{Q}_0$ does not admit undefined terms. The *traditional approach to undefinedness* in mathematical practice is to treat undefined terms as legitimate, nondenoting terms that can be components of meaningful statements. $\mathcal{Q}_0^{\mathrm{u}}$ is a modification of Andrews' type theory $\mathcal{Q}_0$ that directly formalizes the traditional approach to undefinedness. This paper presents $\mathcal{Q}_0^{\mathrm{u}}$ and proves that the proof system of $\mathcal{Q}_0^{\mathrm{u}}$ is sound and complete with respect to its semantics which is based on Henkin-style general models. The paper's development of $\mathcal{Q}_0^{\mathrm{u}}$ closely follows Andrews' development of $\mathcal{Q}_0$ to clearly delineate the differences between the two systems.

1 Introduction

In 1940 Alonzo Church introduced in [4] a version of simple type theory with lambda-notation now known as *Church's type theory*. Church's students Leon Henkin and Peter B. Andrews extensively studied and refined Church's type theory. Henkin proved that Church's type theory is complete with respect to a semantics based on general models [14] and showed that Church's type theory can be reformulated so that it is based on only the primitive notions of function application, function abstraction, equality, and (definite) description [15]. Andrews devised a simple and elegant proof system for Henkin's reformulation of Church's type theory [1]. He also formulated a version of Church's type theory called $\mathcal{Q}_0$ that employs the ideas developed by Church, Henkin, and himself. $\mathcal{Q}_0$ is meticulously described and analyzed in [2] and is the logic of the TPS Theorem Proving System [3].

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Church's type theory has had a profound impact on many areas of computer science, especially programming languages, automated reasoning, formal methods, type theory, and formalized mathematics. It is the fountainhead of a long stream of typed lambda calculi that includes systems such as *System F* [12], *Martin-Löf type theory* [17], and the *Calculus of Constructions* [5]. Several computer theorem proving systems are based on versions of Church's type theory including HOL [13], IMPS [10, 11], Isabelle [19], ProofPower [16], PVS [18], and TPS.

One of the principal virtues of Church's type theory is that it has great expressivity, both theoretical and practical. However, like other traditional logics, Church's type theory assumes that terms are always defined. Despite the fact that undefined terms are commonplace in mathematics (and computer science), undefined terms cannot be directly expressed in Church's type theory—as they are in mathematical practice.

A term is *undefined* if it has no prescribed meaning or if it denotes a value that does not exist.[2] There are two main sources of undefinedness in mathematics. The first source is terms that denote an application of a function. A function f usually has both a *domain of definition* D_f consisting of the values at which it is defined and a *domain of application* D_f^* consisting of the values to which it may be applied. (The domain of definition of a function is usually called simply the *domain* of the function.) These two domains are not always the same, but obviously $D_f \subseteq D_f^*$. A *function application* is a term $f(a)$ that denotes the application of a function f to an argument $a \in D_f^*$. $f(a)$ is *undefined* if $a \notin D_f$. We will say that a function is *partial* if $D_f \neq D_f^*$ and *total* if $D_f = D_f^*$.

The second source of undefinedness is terms that are intended to uniquely describe a value. A *definite description* is a term t of the form "the x that has property P". t is *undefined* if there is no unique x (i.e., none or more than one) that has property P. Definite descriptions are quite common in mathematics but often occur in a disguised form. For example, "the limit of $\sin\frac{1}{x}$ as x approaches 0" is a definite description—which is undefined since the limit does not exist.

There is a *traditional approach to undefinedness* that is widely practiced in mathematics and even taught to some extent to students in secondary school. This approach treats undefined terms as legitimate, nondenoting terms that can be components of meaningful statements. The traditional approach is based on three principles:

1. Atomic terms (i.e., variables and constants) are always defined—they always denote something.

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[2]Some of the text in this section concerning undefinedness is taken from [9].

2. Compound terms may be undefined. A function application $f(a)$ is undefined if f is undefined, a is undefined, or $a \notin D_f$. A definite description "the x that has property P" is undefined if there is no x that has property P or there is more than one x that has property P.

3. Formulas are always true or false, and hence, are always defined. To ensure the definedness of formulas, a function application $p(a)$ formed by applying a predicate p to an argument a is *false* if p is undefined, a is undefined, or $a \notin D_p$.

A logic that formalizes the traditional approach to undefinedness has two advantages over a traditional logic that does not. First, the use of the traditional approach in informal mathematics can be directly formalized, yielding a result that is close to mathematical practice. Second, statements involving partial functions and undefined terms can be expressed very concisely. In particular, assumptions about the definedness of terms and functions often do not have to be made explicit. Concise informal mathematical statements involving partial functions or undefinedness can usually only be expressed in a traditional logic by verbose statements in which definedness assumptions are explicit. For evidence and further discussion of these assertions, see [9].

We presented in [6] a version of Church's type system named PF that formalizes the traditional approach to undefinedness. PF is the basis for LUTINS [7, 8], the logic of the IMPS theorem proving system [10, 11]. The paper [6] includes a proof that PF is complete with respect to a Henkin-style general models semantics. The proof, however, contains a mistake: the tautology theorem does not hold in PF as claimed. This mistake can be corrected by adding modus ponens and a technical axiom schema involving equality to PF's proof system. In [9] we introduced a version of Church's type system with undefinedness called STTwU which is simpler than PF. The proof system of STTwU is claimed to be complete, but a proof of completeness is not given in [9].

The purpose of this paper is to carefully show what changes have to be made to Church's type theory in order to formalize the traditional approach to undefinedness. We do this by presenting a modification of Andrews' type theory $\mathcal{Q}_0$ called $\mathcal{Q}_0^{\mathrm{u}}$. Our goal is to keep $\mathcal{Q}_0^{\mathrm{u}}$ as close to $\mathcal{Q}_0$ as possible, changing as few of the definitions in [2] concerning $\mathcal{Q}_0$ as possible. We present the syntax, semantics and proof system of $\mathcal{Q}_0^{\mathrm{u}}$ and prove that the proof system is sound and complete with respect to its semantics. A series of notes indicates precisely where and how $\mathcal{Q}_0$ and $\mathcal{Q}_0^{\mathrm{u}}$ diverge from each other.

Our presentation of $\mathcal{Q}_0^{\mathrm{u}}$ differs from the presentation of PF in [6] in the following ways:

1. The notation and terminology for $\mathcal{Q}_0^{\mathrm{u}}$ is almost identical to the notation and terminology for $\mathcal{Q}_0$ given in [2] unlike the notation and terminology for PF.

2. The semantics of $\mathcal{Q}_0^{\mathrm{u}}$ is simpler than the semantics of PF.

3. The proof system of $\mathcal{Q}_0^{\mathrm{u}}$ is complete unlike the proof system of PF.

4. The proof of the completeness theorem for $\mathcal{Q}_0^{\mathrm{u}}$ is presented in greater detail than the (erroneous) proof of the completeness theorem for PF.

The paper is organized as follows. The syntax of $\mathcal{Q}_0^{\mathrm{u}}$ is defined in section 2. A Henkin-style general models semantics for $\mathcal{Q}_0^{\mathrm{u}}$ is presented in section 3. Section 4 introduces several important defined logical constants and notational abbreviations. Section 5 gives the proof system of $\mathcal{Q}_0^{\mathrm{u}}$. Some metatheorems of $\mathcal{Q}_0^{\mathrm{u}}$ and the soundness and completeness theorems for $\mathcal{Q}_0^{\mathrm{u}}$ are proved in sections 6 and 7, respectively. The paper ends with a conclusion in section 8.

The great majority of the definitions for $\mathcal{Q}_0^{\mathrm{u}}$ are exactly the same as those for $\mathcal{Q}_0$ given in [2]. We repeat only the most important and least obvious definitions for $\mathcal{Q}_0$; for the others the reader is referred to [2].

2 Syntax of $\mathcal{Q}_0^{\mathrm{u}}$

The syntax of $\mathcal{Q}_0^{\mathrm{u}}$ is almost exactly the same as that of $\mathcal{Q}_0$. The only difference is that just one iota constant is primitive in $\mathcal{Q}_0$, while infinitely many iota constants are primitive in $\mathcal{Q}_0^{\mathrm{u}}$.

A *type symbol* of $\mathcal{Q}_0^{\mathrm{u}}$ is defined inductively as follows:

1. $\imath$ is a type symbol.

2. o is a type symbol.

3. If α and β are type symbols, then $(\alpha\beta)$ is a type symbol.

Let $\mathcal{T}$ denote the set of type symbols. $\alpha, \beta, \gamma, \ldots$ are syntactic variables ranging over type symbols. When there is no loss of meaning, matching pairs of parentheses in type symbols may be omitted. We assume that type combination associates to the left so that a type of the form $((\alpha\beta)\gamma)$ may be written as $\alpha\beta\gamma$.

The *primitive symbols* of $\mathcal{Q}_0^{\mathrm{u}}$ are the following:

1. *Improper symbols*: $[$, $]$, λ.

2. A denumerable set of *variables* of type α for each $\alpha \in \mathcal{T}$: f_α, g_α, h_α, x_α, y_α, z_α, f_α^1, g_α^1, h_α^1, x_α^1, y_α^1, z_α^1, $\ldots$.

3. *Logical constants*: $\mathsf{Q}_{((o\alpha)\alpha)}$ for each $\alpha \in \mathcal{T}$ and $\iota_{(\alpha(o\alpha))}$ for each $\alpha \in \mathcal{T}$ with $\alpha \neq o$.

4. An unspecified set of *nonlogical constants* of various types.

$\mathbf{x}_\alpha, \mathbf{y}_\alpha, \mathbf{z}_\alpha, \mathbf{f}_\alpha, \mathbf{g}_\alpha, \mathbf{h}_\alpha, \ldots$ are syntactic variables ranging over variables of type α.

Note 1 (Iota Constants). Only $\iota_{\iota(o\iota)}$ is a primitive logical constant in $\mathcal{Q}_0$; each other $\iota_{\alpha(o\alpha)}$ is a nonprimitive logical constant in $\mathcal{Q}_0$ defined according to an inductive scheme presented by Church in [4] (see [2, pp. 233–4]). We will see in the next section that the iota constants have a different semantics in $\mathcal{Q}_0^{\mathrm{u}}$ than in $\mathcal{Q}_0$. As a result, it is not possible to define the iota constants in $\mathcal{Q}_0^{\mathrm{u}}$ as they are defined in $\mathcal{Q}_0$, and thus they must be primitive in $\mathcal{Q}_0^{\mathrm{u}}$. Notice that $\iota_{o(oo)}$ is not a primitive logical constant of $\mathcal{Q}_0^{\mathrm{u}}$. It has been left out because it serves no useful purpose. It can be defined as a nonprimitive logical constant as in [2, p. 233] if desired. ∎

We are now ready to define a *wff of type α (wff$_\alpha$)*. $\mathbf{A}_\alpha, \mathbf{B}_\alpha, \mathbf{C}_\alpha, \ldots$ are syntactic variables ranging over wffs of type α. A *wff$_\alpha$* is then defined inductively as follows:

1. A variable or primitive constant of type α is a wff$_\alpha$.

2. $[\mathbf{A}_{\alpha\beta}\mathbf{B}_\beta]$ is a wff$_\alpha$.

3. $[\lambda \mathbf{x}_\beta \mathbf{A}_\alpha]$ is a wff$_{\alpha\beta}$.

A wff of the form $[\mathbf{A}_{\alpha\beta}\mathbf{B}_\beta]$ is called a *function application* and a wff of the form $[\lambda \mathbf{x}_\beta \mathbf{A}_\alpha]$ is called a *function abstraction*. When there is no loss of meaning, matching pairs of square brackets in wffs may be omitted. We assume that wff combination of the form $[\mathbf{A}_{\alpha\beta}\mathbf{B}_\beta]$ associates to the left so that a wff $[[\mathbf{C}_{\gamma\beta\alpha}\mathbf{A}_\alpha]\mathbf{B}_\beta]$ may be written as $\mathbf{C}_{\gamma\beta\alpha}\mathbf{A}_\alpha\mathbf{B}_\beta$.

3 Semantics of $\mathcal{Q}_0^{\mathrm{u}}$

The traditional approach to definedness is formalized in $\mathcal{Q}_0^{\mathrm{u}}$ by modifying the semantics of $\mathcal{Q}_0$. Two principal changes are made to the $\mathcal{Q}_0$ semantics: (1) The notion of a general model is redefined to include partial functions as well as total functions. (2) The valuation function for wffs is made into a partial function that assigns a value to a wff iff the wff is defined according to the traditional approach.

A *frame* is a collection $\{\mathcal{D}_\alpha \mid \alpha \in \mathcal{T}\}$ of nonempty domains such that:

1. $\mathcal{D}_o = \{\mathsf{T}, \mathsf{F}\}$.

2. For $\alpha, \beta \in \mathcal{T}$, $\mathcal{D}_{\alpha\beta}$ is some set of *total* functions from $\mathcal{D}_\beta$ to $\mathcal{D}_\alpha$ if $\alpha = o$ and is some set of *partial and total* functions from $\mathcal{D}_\beta$ to $\mathcal{D}_\alpha$ if $\alpha \neq o$.[3]

$\mathcal{D}_o$ is the *domain of truth values*, $\mathcal{D}_\iota$ is the *domain of individuals*, and for $\alpha, \beta \in \mathcal{T}$, $\mathcal{D}_{\alpha\beta}$ is a *function domain*. For all $\alpha \in \mathcal{T}$, the *identity relation* on $\mathcal{D}_\alpha$ is the total function $q \in \mathcal{D}_{o\alpha\alpha}$ such that, for all $x, y \in \mathcal{D}_\alpha$, $q(x)(y) = \mathsf{T}$ iff $x = y$. For all $\alpha \in \mathcal{T}$ with $\alpha \neq o$, the *unique member selector* on $\mathcal{D}_\alpha$ is the partial function $f \in \mathcal{D}_{\alpha(o\alpha)}$ such that, for all $s \in \mathcal{D}_{o\alpha}$, if the predicate s represents a singleton $\{x\} \subseteq \mathcal{D}_\alpha$, then $f(s) = x$, and otherwise $f(s)$ is undefined.

Note 2 (Function Domains). In a $\mathcal{Q}_0$ frame a function domain $\mathcal{D}_{\alpha\beta}$ contains only total functions, while in a $\mathcal{Q}_0^{\mathrm{u}}$ frame a function domain $\mathcal{D}_{o\beta}$ contains only total functions but a function domain $\mathcal{D}_{\alpha\beta}$ with $\alpha \neq o$ contains partial functions as well as total functions. ∎

An *interpretation* $\langle \{\mathcal{D}_\alpha \mid \alpha \in \mathcal{T}\}, \mathcal{J} \rangle$ of $\mathcal{Q}_0^{\mathrm{u}}$ consists of a frame and a function $\mathcal{J}$ that maps each primitive constant of $\mathcal{Q}_0^{\mathrm{u}}$ of type α to an element of $\mathcal{D}_\alpha$ such that $\mathcal{J}(\mathsf{Q}_{o\alpha\alpha})$ is the identity relation on $\mathcal{D}_\alpha$ for each $\alpha \in \mathcal{T}$ and $\mathcal{J}(\iota_{\alpha(o\alpha)})$ is the unique member selector on $\mathcal{D}_\alpha$ for each $\alpha \in \mathcal{T}$ with $\alpha \neq o$.

Note 3 (Definite Description Operators). The $\iota_{\alpha(o\alpha)}$ in $\mathcal{Q}_0$ are *description operators*: if $\mathbf{A}_{o\alpha}$ denotes a singleton, then the value of $\iota_{\alpha(o\alpha)}\mathbf{A}_{o\alpha}$ is the unique member of the singleton, and otherwise the value of $\iota_{\alpha(o\alpha)}\mathbf{A}_{o\alpha}$ is *unspecified*. In contrast, the $\iota_{\alpha(o\alpha)}$ in $\mathcal{Q}_0^{\mathrm{u}}$ are *definite description operators*: if $\mathbf{A}_{o\alpha}$ denotes a singleton, then the value of $\iota_{\alpha(o\alpha)}\mathbf{A}_{o\alpha}$ is the unique member of the singleton, and otherwise the value of $\iota_{\alpha(o\alpha)}\mathbf{A}_{o\alpha}$ is *undefined*. ∎

An *assignment* into a frame $\{\mathcal{D}_\alpha \mid \alpha \in \mathcal{T}\}$ is a function φ whose domain is the set of variables of $\mathcal{Q}_0^{\mathrm{u}}$ such that, for each variable $\mathbf{x}_\alpha$, $\varphi(\mathbf{x}_\alpha) \in \mathcal{D}_\alpha$. Given an assignment φ, a variable $\mathbf{x}_\alpha$, and $d \in \mathcal{D}_\alpha$, let $(\varphi : \mathbf{x}_\alpha/d)$ be the assignment ψ such that $\psi(\mathbf{x}_\alpha) = d$ and $\psi(\mathbf{y}_\beta) = \varphi(\mathbf{y}_\beta)$ for all variables $\mathbf{y}_\beta \neq \mathbf{x}_\alpha$.

An interpretation $\mathcal{M} = \langle \{\mathcal{D}_\alpha \mid \alpha \in \mathcal{T}\}, \mathcal{J} \rangle$ is a *general model* for $\mathcal{Q}_0^{\mathrm{u}}$ if there is a binary function $\mathcal{V}^{\mathcal{M}}$ such that, for each assignment φ and wff $\mathbf{C}_\gamma$, either $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{C}_\gamma) \in \mathcal{D}_\gamma$ or $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{C}_\gamma)$ is undefined and the following conditions are satisfied for all assignments φ and all wffs $\mathbf{C}_\gamma$:

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[3] The condition that a domain $D_{o\beta}$ contains only total functions is needed to ensure that the law of extensionality holds for predicates. This condition is weaker than the condition used in the semantics for PF [6] and its extended versions PF* [7] and LUTINS[7, 8]. In these logics, a domain D_γ contains only total functions iff γ has the form $o\beta_1 \cdots \beta_n$ where $n \geq 1$. The weaker condition, which is due to Aaron Stump [20], yields a semantics that is simpler.

(a) Let $\mathbf{C}_\gamma$ be a variable of $\mathcal{Q}_0^u$. Then $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{C}_\gamma) = \varphi(\mathbf{C}_\gamma)$.

(b) Let $\mathbf{C}_\gamma$ be a primitive constant of $\mathcal{Q}_0^u$. Then $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{C}_\gamma) = \mathcal{J}(\mathbf{C}_\gamma)$.

(c) Let $\mathbf{C}_\gamma$ be $[\mathbf{A}_{\alpha\beta}\mathbf{B}_\beta]$. If $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_{\alpha\beta})$ is defined, $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_\beta)$ is defined, and the function $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_{\alpha\beta})$ is defined at the argument $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_\beta)$, then

$$\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{C}_\gamma) = \mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_{\alpha\beta})(\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_\beta)),$$

the value of the function $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_{\alpha\beta})$ at the argument $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{B}_\beta)$. Otherwise, $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{C}_\gamma) = \mathsf{F}$ if $\alpha = o$ and $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{C}_\gamma)$ is undefined if $\alpha \neq o$.

(d) Let $\mathbf{C}_\gamma$ be $[\lambda\mathbf{x}_\beta\mathbf{B}_\alpha]$. Then $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{C}_\gamma)$ is the (partial or total) function $f \in \mathcal{D}_{\alpha\beta}$ such that, for each $d \in \mathcal{D}_\beta$, $f(d) = \mathcal{V}_{(\varphi:\mathbf{x}_\beta/d)}^{\mathcal{M}}(\mathbf{B}_\alpha)$ if $\mathcal{V}_{(\varphi:\mathbf{x}_\alpha/d)}^{\mathcal{M}}(\mathbf{B}_\alpha)$ is defined and $f(d)$ is undefined if $\mathcal{V}_{(\varphi:\mathbf{x}_\beta/d)}^{\mathcal{M}}(\mathbf{B}_\alpha)$ is undefined.

Note 4 (Valuation Function). In $\mathcal{Q}_0$, if $\mathcal{M}$ is a general model, then $\mathcal{V}^{\mathcal{M}}$ is total and the value of $\mathcal{V}^{\mathcal{M}}$ on a function abstraction is always a total function. In $\mathcal{Q}_0^u$, if $\mathcal{M}$ is a general model, then $\mathcal{V}^{\mathcal{M}}$ is partial and the value of $\mathcal{V}^{\mathcal{M}}$ on a function abstraction can be either a partial or a total function. ∎

PROPOSITION 1. *Let $\mathcal{M}$ be a general model for $\mathcal{Q}_0^u$. Then $\mathcal{V}^{\mathcal{M}}$ is defined on all variables, primitive constants, function abstractions, and function applications of type o and is defined on only a proper subset of function applications of type $\alpha \neq o$.*

Note 5 (Traditional Approach). $\mathcal{Q}_0^u$ clearly satisfies the three principles of the traditional approach to undefinedness. Like other traditional logics, $\mathcal{Q}_0$ only satisfies the first principle. ∎

Let $\mathcal{H}$ be a set of wffs$_o$ and $\mathcal{M}$ be a general model for $\mathcal{Q}_0^u$. $\mathbf{A}_o$ is *valid* in $\mathcal{M}$, written $\mathcal{M} \models \mathbf{A}_o$, if $\mathcal{V}_\varphi^{\mathcal{M}}(\mathbf{A}_o) = \mathsf{T}$ for all assignments φ. $\mathcal{M}$ is a *general model* for $\mathcal{H}$ if $\mathcal{M} \models \mathbf{B}_o$ for all $\mathbf{B}_o \in \mathcal{H}$. $\mathbf{A}_o$ is *valid (in the general sense)* in $\mathcal{H}$, written $\mathcal{H} \models \mathbf{A}_o$, if $\mathcal{M} \models \mathbf{A}_o$ for every general model $\mathcal{M}$ for $\mathcal{H}$. $\mathbf{A}_o$ is *valid (in the general sense)* in $\mathcal{Q}_0^u$, written $\models \mathbf{A}_o$, if $\emptyset \models \mathbf{A}_o$.

Note 6 (Mutual Interpretability). $\mathcal{Q}_0^u$ can be interpreted in $\mathcal{Q}_0$ by viewing a function of type $\alpha\beta$ in $\mathcal{Q}_0^u$ as a function (predicate) of type $o\alpha\beta$

in $\mathcal{Q}_0$. $\mathcal{Q}_0$ can be interpreted in $\mathcal{Q}_0^{\mathrm{u}}$ by viewing a function of type $\alpha\beta$ in $\mathcal{Q}_0$ as a total function of type $\alpha\beta$ in $\mathcal{Q}_0^{\mathrm{u}}$. Thus $\mathcal{Q}_0$ and $\mathcal{Q}_0^{\mathrm{u}}$ are equivalent in the sense of being mutually interpretable. ∎

4   Definitions and Abbreviations

As Andrews does in [2, p. 212], we will introduce several defined logical constants and notational abbreviations. The former includes constants for true and false, the propositional connectives, and a canonical undefined wff. The latter includes notation for equality, the propositional connectives, universal and existential quantification, defined and undefined wffs, quasi-equality, and definite description.

$[\mathbf{A}_\alpha = \mathbf{B}_\alpha]$	stands for	$[\mathsf{Q}_{o\alpha\alpha}\mathbf{A}_\alpha\mathbf{B}_\alpha]$.
T_o	stands for	$[\mathsf{Q}_{ooo} = \mathsf{Q}_{ooo}]$.
F_o	stands for	$[\lambda x_o T_o] = [\lambda x_o x_o]$.
$[\forall \mathbf{x}_\alpha \mathbf{A}_o]$	stands for	$[\lambda y_\alpha T_o] = [\lambda \mathbf{x}_\alpha \mathbf{A}_o]$.
$\wedge_{ooo}$	stands for	$[\lambda x_o \lambda y_o [[\lambda g_{ooo} [g_{ooo} T_o T_o]] = [\lambda g_{ooo} [g_{ooo} x_o y_o]]]]$.
$[\mathbf{A}_o \wedge \mathbf{B}_o]$	stands for	$[\wedge_{ooo} \mathbf{A}_o \mathbf{B}_o]$.
$\supset_{ooo}$	stands for	$[\lambda x_o \lambda y_o [x_o = [x_o \wedge y_o]]]$.
$[\mathbf{A}_o \supset \mathbf{B}_o]$	stands for	$[\supset_{ooo} \mathbf{A}_o \mathbf{B}_o]$.
$\sim_{oo}$	stands for	$[\mathsf{Q}_{ooo} F_o]$.
$[\sim\mathbf{A}_o]$	stands for	$[\sim_{oo} \mathbf{A}_o]$.
$\vee_{ooo}$	stands for	$[\lambda x_o \lambda y_o [\sim[[\sim x_o] \wedge [\sim y_o]]]]$.
$[\mathbf{A}_o \vee \mathbf{B}_o]$	stands for	$[\vee_{ooo} \mathbf{A}_o \mathbf{B}_o]$.
$[\exists \mathbf{x}_\alpha \mathbf{A}_o]$	stands for	$[\sim[\forall \mathbf{x}_\alpha \sim\mathbf{A}_o]]$.
$[\exists_1 \mathbf{x}_\alpha \mathbf{A}_o]$	stands for	$[\exists y_\alpha [[\lambda \mathbf{x}_\alpha \mathbf{A}_o] = \mathsf{Q}_{o\alpha\alpha} y_\alpha]]$ where y_α does not occur in $\mathbf{A}_o$.
$[\mathbf{A}_\alpha \neq \mathbf{B}_\alpha]$	stands for	$[\sim[\mathbf{A}_\alpha = \mathbf{B}_\alpha]]$.
$[\mathbf{A}_\alpha \downarrow]$	stands for	$[\exists x_\alpha [x_\alpha = \mathbf{A}_\alpha]]$ where x_α does not occur in $\mathbf{A}_\alpha$.
$[\mathbf{A}_\alpha \uparrow]$	stands for	$[\sim[\mathbf{A}_\alpha \downarrow]]$
$[\mathbf{A}_\alpha \simeq \mathbf{B}_\alpha]$	stands for	$[\mathbf{A}_\alpha \downarrow \vee \mathbf{B}_\alpha \downarrow] \supset [\mathbf{A}_\alpha = \mathbf{B}_\alpha]$.
$[\mathrm{I}\mathbf{x}_\alpha \mathbf{A}_o]$	stands for	$[\iota_{\alpha(o\alpha)} [\lambda \mathbf{x}_\alpha \mathbf{A}_o]]$.
$\perp_\alpha$	stands for	$[\mathrm{I} x_\alpha [x_\alpha \neq x_\alpha]]$ where $\alpha \neq o$.

$[\exists_1 \mathbf{x}_\alpha \mathbf{A}_o]$ asserts that there is a unique $\mathbf{x}_\alpha$ that satisfies $\mathbf{A}_o$.

$[\mathrm{I}\mathbf{x}_\alpha \mathbf{A}_o]$ is called a *definite description*. It denotes the unique $\mathbf{x}_\alpha$ that satisfies $\mathbf{A}_o$. If there is no or more than one such $\mathbf{x}_\alpha$, it is undefined. Following Bertrand Russell and Church, Andrews denotes this definite description operator as an inverted lower case iota (ι). We represent this operator by an (inverted) capital iota (I).

$[\mathbf{A}_\alpha\!\downarrow]$ says that $\mathbf{A}_\alpha$ is defined, and similarly, $[\mathbf{A}_\alpha\!\uparrow]$ says that $\mathbf{A}_\alpha$ is undefined. $[\mathbf{A}_\alpha \simeq \mathbf{B}_\alpha]$ says that $\mathbf{A}_\alpha$ and $\mathbf{B}_\alpha$ are *quasi-equal*, i.e., that $\mathbf{A}_\alpha$ and $\mathbf{B}_\alpha$ are either both defined and equal or both undefined. $\bot_\alpha$ is a canonical undefined wff of type α.

Note 7 (Definedness Notation). In $\mathcal{Q}_0$, $[\mathbf{A}_\alpha\!\downarrow]$ is always true, $[\mathbf{A}_\alpha\!\uparrow]$ is always false, $[\mathbf{A}_\alpha \simeq \mathbf{B}_\alpha]$ is always equal to $[\mathbf{A}_\alpha = \mathbf{B}_\alpha]$, and $\bot_\alpha$ denotes an unspecified value. ∎

5 Proof System of $\mathcal{Q}_0^{\mathrm{u}}$

In this section we present the proof system of $\mathcal{Q}_0^{\mathrm{u}}$ which is derived from the proof system of $\mathcal{Q}_0$. The issue of definedness makes the proof system of $\mathcal{Q}_0^{\mathrm{u}}$ moderately more complicated than the proof system for $\mathcal{Q}_0$. While $\mathcal{Q}_0$ has only five axiom schemas and one rule of inference, $\mathcal{Q}_0^{\mathrm{u}}$ has the following thirteen axiom schemas and two rules of inference:

A1 (Truth Values)

$$[g_{oo}T_o \wedge g_{oo}F_o] = \forall x_o[g_{oo}x_o].$$

A2 (Leibniz' Law)

$$[x_\alpha = y_\alpha] \supset [h_{o\alpha}x_\alpha = h_{o\alpha}y_\alpha].$$

A3 (Extensionality)

$$[f_{\alpha\beta} = g_{\alpha\beta}] = \forall x_\beta[f_{\alpha\beta}x_\beta \simeq g_{\alpha\beta}x_\beta].$$

A4 (Beta-Reduction)

$$\mathbf{A}_\alpha\!\downarrow \,\supset [[\lambda\mathbf{x}_\alpha\mathbf{B}_\beta]\mathbf{A}_\alpha \simeq \mathsf{S}_{\mathbf{A}_\alpha}^{\mathbf{x}_\alpha}\mathbf{B}_\beta]$$

provided $\mathbf{A}_\alpha$ is free for $\mathbf{x}_\alpha$ in $\mathbf{B}_\beta$.[4]

A5 (Variables are Defined)

$$\mathbf{x}_\alpha\!\downarrow.$$

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[4]$\mathsf{S}_{\mathbf{A}_\alpha}^{\mathbf{x}_\alpha}\mathbf{B}_\beta$ is the result of substituting $\mathbf{A}_\alpha$ for each free occurrence of $\mathbf{x}_\alpha$ in $\mathbf{B}_\beta$.

A6 (Primitive Constants are Defined)

$\mathbf{c}_\alpha \downarrow$ where $\mathbf{c}_\alpha$ is a primitive constant.[5]

A7 (Function Abstractions are Defined)

$[\lambda \mathbf{x}_\alpha \mathbf{B}_\beta] \downarrow$.

A8 (Function Applications of Type o are Defined)

$\mathbf{A}_{o\beta} \mathbf{B}_\beta \downarrow$.

A9 (Improper Function Application of Type o)

$[\mathbf{A}_{o\beta}\uparrow \vee \mathbf{B}_\beta\uparrow] \supset \sim[\mathbf{A}_{o\beta}\mathbf{B}_\beta]$.

A10 (Improper Function Application of Type $\alpha \neq o$)

$[\mathbf{A}_{\alpha\beta}\uparrow \vee \mathbf{B}_\beta\uparrow] \supset \mathbf{A}_{\alpha\beta}\mathbf{B}_\beta\uparrow$ where $\alpha \neq o$.

A11 (Equality and Quasi-Quality)

$\mathbf{A}_\alpha \downarrow \supset [\mathbf{B}_\alpha \downarrow \supset [[\mathbf{A}_\alpha \simeq \mathbf{B}_\alpha] \simeq [\mathbf{A}_\alpha = \mathbf{B}_\alpha]]]$.

A12 (Proper Definite Description)

$\exists_1 \mathbf{x}_\alpha \mathbf{A}_o \supset [[\mathrm{I}\mathbf{x}_\alpha \mathbf{A}_o]\downarrow \wedge \mathsf{S}^{\mathbf{x}_\alpha}_{[\mathrm{I}\mathbf{x}_\alpha \mathbf{A}_o]}\mathbf{A}_o]$ where $\alpha \neq o$

and provided $\mathrm{I}\mathbf{x}_\alpha \mathbf{A}_o$ is free for $\mathbf{x}_\alpha$ in $\mathbf{A}_o$.

A13 (Improper Definite Description)

$\sim[\exists_1 \mathbf{x}_\alpha \mathbf{A}_o] \supset [\mathrm{I}\mathbf{x}_\alpha \mathbf{A}_o]\uparrow$ where $\alpha \neq o$.

R1 (Quasi-Equality Substitution) From $\mathbf{A}_\alpha \simeq \mathbf{B}_\alpha$ and $\mathbf{C}_o$ infer the result of replacing one occurrence of $\mathbf{A}_\alpha$ in $\mathbf{C}_o$ by an occurrence of $\mathbf{B}_\alpha$, provided that the occurrence of $\mathbf{A}_\alpha$ in $\mathbf{C}_o$ is not (an occurrence of a variable) immediately preceded by λ.

R2 (Modus Ponens) From $\mathbf{A}_o$ and $\mathbf{A}_o \supset \mathbf{B}_o$ infer $\mathbf{B}_o$.

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[5]Notice that $\mathbf{c}_\alpha \downarrow$ is false if $\mathbf{c}_\alpha$ is a defined constant $\perp_\alpha$ where $\alpha \neq o$.

Note 8 (Axiom Schemas). The axiom schemas A1, A2, A3, A4, and A12 of $\mathcal{Q}_0^u$ correspond to the five axiom schemas of $\mathcal{Q}_0$. A1 and A2 are exactly the same as the first and second axiom schemas of $\mathcal{Q}_0$. A3 and A4 are modifications of the third and fourth axiom schemas of $\mathcal{Q}_0$. A3 is the axiom of extensionality for partial and total functions, and A4 is beta-reduction for functions that may be partial and arguments that may be undefined.

The seven axiom schemas A5–A11 of $\mathcal{Q}_0^u$ deal with the definedness of wffs. A5 and A6 address the first principle of the traditional approach to undefinedness, A10 addresses the second principle, and A8 and A9 address the third principle. A7 states that a function abstraction always denotes some function, either partial or total. And A11 is a technical axiom schema for identifying equality with quasi-equality when applied to defined wffs.

The last two axiom schemas of $\mathcal{Q}_0^u$ state the properties of definite descriptions. A12 states that proper definite descriptions are defined and denote the unique value satisfying the description; it corresponds to the fifth axiom schema of $\mathcal{Q}_0$. A13 states that improper definite descriptions are undefined. The proof system of $\mathcal{Q}_0$ leaves improper definite descriptions unspecified. ∎

Note 9 (Rules of Inference). $\mathcal{Q}_0^u$'s R1 rule of inference, Quasi-Equality Substitution, corresponds to $\mathcal{Q}_0$'s single rule of inference, which is equality substitution. These rules are exactly the same except that the $\mathcal{Q}_0^u$ rule requires only *quasi-equality* ($\simeq$) between the target wff and the substitution wff, while the $\mathcal{Q}_0$ rule requires *equality* ($=$).

$\mathcal{Q}_0^u$'s R2 rule of inference, Modus Ponens, is a primitive rule of inference, but modus ponens is a derived rule of inference in $\mathcal{Q}_0$. Modus ponens must be primitive in $\mathcal{Q}_0^u$ since it is needed to discharge the definedness conditions on instances of A4, the schema for beta-reduction, and A11. ∎

A *proof* of a wff$_o$ $\mathbf{A}_o$ in $\mathcal{Q}_0^u$ is a finite sequence of wffs$_o$, ending with $\mathbf{A}_o$, such that each member in the sequence is an instance of an axiom schema of $\mathcal{Q}_0^u$ or is inferred from preceding members in the sequence by a rule of inference of $\mathcal{Q}_0^u$. A *theorem* of $\mathcal{Q}_0^u$ is a wff$_o$ for which there is a proof in $\mathcal{Q}_0^u$.

Let $\mathcal{H}$ be a set of wffs$_o$. A *proof of $\mathbf{A}_o$ from $\mathcal{H}$* in $\mathcal{Q}_0^u$ consists of two finite sequences $\mathcal{S}_1$ and $\mathcal{S}_2$ of wffs$_o$ such that $\mathcal{S}_1$ is a proof in $\mathcal{Q}_0^u$, $\mathbf{A}_o$ is the last member of $\mathcal{S}_2$, and each member $\mathbf{D}_o$ of $\mathcal{S}_2$ satisfies at least one of the following conditions:

1. $\mathbf{D}_o \in \mathcal{H}$.

2. $\mathbf{D}_o$ is a member of $\mathcal{S}_1$ (and hence a theorem of $\mathcal{Q}_0^u$).

3. $\mathbf{D}_o$ is inferred from two preceding members $\mathbf{A}_\alpha \simeq \mathbf{B}_\alpha$ and $\mathbf{C}_o$ of $\mathcal{S}_2$ by R1, provided that the occurrence of $\mathbf{A}_\alpha$ in $\mathbf{C}_o$ is not in a well-formed

part $\lambda \mathbf{x}_\beta \mathbf{E}_\gamma$ of $\mathbf{C}_o$ where $\mathbf{x}_\beta$ is free in a member of $\mathcal{H}$ and free in $\mathbf{A}_\alpha \simeq \mathbf{B}_\beta$.

4. $\mathbf{D}_o$ is inferred from two preceding members of $\mathcal{S}_2$ by R2.

We write $\mathcal{H} \vdash \mathbf{A}_o$ to mean there is a proof of $\mathbf{A}_o$ from $\mathcal{H}$ in $\mathcal{Q}_0^{\mathrm{u}}$. $\vdash \mathbf{A}_o$ is written instead of $\emptyset \vdash \mathbf{A}_o$. Clearly, $\mathbf{A}_o$ is a theorem of $\mathcal{Q}_0^{\mathrm{u}}$ iff $\vdash \mathbf{A}_o$.

The next two theorems follow immediately from the definition above.

THEOREM 2 (R1′). *If $\mathcal{H} \vdash \mathbf{A}_\alpha \simeq \mathbf{B}_\alpha$ and $\mathcal{H} \vdash \mathbf{C}_o$, then $\mathcal{H} \vdash \mathbf{D}_o$, where $\mathbf{D}_o$ is the result of replacing one occurrence of $\mathbf{A}_\alpha$ in $\mathbf{C}_o$ by an occurrence of $\mathbf{B}_\alpha$, provided that the occurrence of $\mathbf{A}_\alpha$ in $\mathbf{C}_o$ is not immediately preceded by λ or in a well-formed part $\lambda \mathbf{x}_\beta \mathbf{E}_\gamma$ of $\mathbf{C}_o$ where $\mathbf{x}_\beta$ is free in a member of $\mathcal{H}$ and free in $\mathbf{A}_\alpha \simeq \mathbf{B}_\alpha$.*

THEOREM 3 (R2′). *If $\mathcal{H} \vdash \mathbf{A}_o$ and $\mathcal{H} \vdash \mathbf{A}_o \supset \mathbf{B}_o$, then $\mathcal{H} \vdash \mathbf{B}_o$.*

6 Some Metatheorems

In this section we prove some metatheorems of $\mathcal{Q}_0^{\mathrm{u}}$ that are needed to prove the soundness and completeness of the proof system of $\mathcal{Q}_0^{\mathrm{u}}$.

PROPOSITION 4 (Wffs of type o are defined). $\vdash \mathbf{A}_o \downarrow$ *for all wffs $\mathbf{A}_o$.*

Proof. Directly implied by axiom schemas A5, A6, and A8. ∎

THEOREM 5 (Beta-Reduction Rule). *If $\mathcal{H} \vdash \mathbf{A}_\alpha \downarrow$ and $\mathcal{H} \vdash \mathbf{C}_o$, then $\mathcal{H} \vdash \mathbf{D}_o$, where $\mathbf{D}_o$ is the result of replacing one occurrence of $[\lambda \mathbf{x}_\alpha \mathbf{B}_\beta] \mathbf{A}_\alpha$ in $\mathbf{C}_o$ by an occurrence of $\mathsf{S}_{\mathbf{A}_\alpha}^{\mathbf{x}_\alpha} \mathbf{B}_\beta$, provided $\mathbf{A}_\alpha$ is free for $\mathbf{x}_\alpha$ in $\mathbf{B}_\beta$ and the occurrence of $[\lambda \mathbf{x}_\alpha \mathbf{B}_\beta] \mathbf{A}_\alpha$ is not in a well-formed part $\lambda \mathbf{y}_\gamma \mathbf{E}_\delta$ of $\mathbf{C}_o$ where $\mathbf{y}_\gamma$ is free in a member of $\mathcal{H}$ and free in $[\lambda \mathbf{x}_\alpha \mathbf{B}_\beta] \mathbf{A}_\alpha$.*

Proof. Follows immediately from A4, R1′, and R2′. ∎

LEMMA 6. $\vdash \mathbf{A}_\alpha \simeq \mathbf{A}_\alpha$.

Proof. Let $\mathbf{x}_\alpha$ be a variable that does not occur in $\mathbf{A}_\alpha$. Then $\mathbf{x}_\alpha \downarrow$ is an instance of A5, and $\mathbf{x}_\alpha \downarrow \supset [[\lambda \mathbf{x}_\alpha \mathbf{A}_\alpha] \mathbf{x}_\alpha \simeq \mathbf{A}_\alpha]$ is an instance of A4. By applying R2′ to these two wffs we obtain $\vdash [[\lambda \mathbf{x}_\alpha \mathbf{A}_\alpha] \mathbf{x}_\alpha \simeq \mathbf{A}_\alpha]$. The conclusion of the lemma then follows by the Beta-Reduction Rule. ∎

LEMMA 7. *If $\mathcal{H} \vdash \mathbf{A}_\alpha \downarrow$ and $\mathcal{H} \vdash \mathbf{B}_\alpha \downarrow$, then $\mathcal{H} \vdash \mathbf{A}_\alpha \simeq \mathbf{B}_\alpha$ iff $\mathcal{H} \vdash \mathbf{A}_\alpha = \mathbf{B}_\alpha$.*

Proof.

($\Rightarrow$): Follows immediately from A11, R1$'$, and R2$'$.

($\Leftarrow$): $\mathcal{H} \vdash [\mathbf{A}_\alpha \simeq \mathbf{B}_\alpha] \simeq [\mathbf{A}_\alpha = \mathbf{B}_\alpha]$ by the first two hypotheses, A11, and R2$'$. $\vdash [\mathbf{A}_\alpha \simeq \mathbf{B}_\alpha] \simeq [\mathbf{A}_\alpha \simeq \mathbf{B}_\alpha]$ by Lemma 6. We obtain $\mathcal{H} \vdash [\mathbf{A}_\alpha = \mathbf{B}_\alpha] \simeq [\mathbf{A}_\alpha \simeq \mathbf{B}_\alpha]$ by applying R1$'$ to these two statements. The conclusion of the lemma then follows by applying R1$'$ to this last statement and $\mathcal{H} \vdash \mathbf{A}_\alpha = \mathbf{B}_\alpha$. ∎

COROLLARY 8. *If* $\vdash \boldsymbol{A}_\alpha \!\downarrow$, *then* $\vdash \boldsymbol{A}_\alpha = \boldsymbol{A}_\alpha$.

Proof. By Lemmas 6 and 7. ∎

LEMMA 9. *If* $\vdash \boldsymbol{A}_\alpha \!\downarrow$ *and* $\vdash \boldsymbol{B}_\beta \simeq \boldsymbol{C}_\beta$, *then* $\vdash \mathsf{S}^{\mathbf{x}_\alpha}_{\mathbf{A}_\alpha}[\boldsymbol{B}_\beta \simeq \boldsymbol{C}_\beta]$, *provided* $\boldsymbol{A}_\alpha$ *is free for* $\boldsymbol{x}_\alpha$ *in* $\boldsymbol{B}_\beta \simeq \boldsymbol{C}_\beta$.

Proof. Follows from Lemma 6 and the Beta-Reduction Rule in a way that is similar to the proof of theorem 5209 in [2]. ∎

COROLLARY 10. *If* $\vdash \boldsymbol{A}_\alpha \!\downarrow$ *and* $\vdash \boldsymbol{B}_o = \boldsymbol{C}_o$, *then* $\vdash \mathsf{S}^{\mathbf{x}_\alpha}_{\mathbf{A}_\alpha}[\boldsymbol{B}_o = \boldsymbol{C}_o]$, *provided* $\boldsymbol{A}_\alpha$ *is free for* $\boldsymbol{x}_\alpha$ *in* $\boldsymbol{B}_o = \boldsymbol{C}_o$.

Proof. By Proposition 4, Lemma 7, and Lemma 9. ∎

LEMMA 11. *If* $\vdash \boldsymbol{B}_\beta \!\downarrow$, *then* $\vdash T_o = [\boldsymbol{B}_\beta = \boldsymbol{B}_\beta]$.

Proof. The proof of $\vdash T_o = [\mathbf{B}_\beta \simeq \mathbf{B}_\beta]$ is similar to the proof of theorem 5210 in [2] with Corollaries 8 and 10 used in place of theorems 5200 and 5209, respectively. The lemma then follows from A11, R1, and R2. ∎

LEMMA 12. *If* $\vdash \boldsymbol{A}_o = \boldsymbol{B}_o$ *and* $\vdash \boldsymbol{C}_o = \boldsymbol{D}_o$, *then* $\vdash [\boldsymbol{A}_o = \boldsymbol{B}_o] \wedge [\boldsymbol{C}_o = \boldsymbol{D}_o]$.

Proof. Similar to the proof of theorem 5213 in [2] with Lemma 11 used in place of theorem 5210. ∎

The proofs of the next four theorems are similar to the proofs of theorems 5215, 5220, 5234, and 5240 except that:

1. Rule R1 and Lemma 7 are used in place of rule R.

2. Rule R1$'$ and Lemma 7 are used in place of rule R$'$.

3. The Beta-Reduction Rule is used in place of the β-Contraction rule.

4. Corollary 8 is used in place of theorem 5200.

5. Axiom schema A4 and Lemma 7 are used in place of theorem 5207.

6. Corollary 10 is used in place of theorem 5209.

7. Lemma 11 is used in place of theorem 5210.

8. Lemma 12 is used in place of theorem 5213.

9. Rule R2$'$ is used in place of theorem 5224 (MP).

10. Axiom schemas A5–A8 are used to discharge definedness conditions.

THEOREM 13 (Universal Instantiation). *If $\mathcal{H} \vdash A_\alpha\!\downarrow$ and $\mathcal{H} \vdash \forall x_\alpha B_o$, then $\mathcal{H} \vdash \mathsf{S}^{x_\alpha}_{A_\alpha} B_o$, provided A_α is free for x_α in B_o.*

Proof. Similar to the proof of theorem 5215 ($\forall$I) in [2]. See the comment above. ■

THEOREM 14 (Universal Generalization). *If $\mathcal{H} \vdash A_o$, then $\mathcal{H} \vdash \forall x_\alpha A_o$, provided x_α is not free in any wff in $\mathcal{H}$.*

Proof. Similar to the proof of theorem 5220 (Gen) in [2]. See the comment above. ■

THEOREM 15 (Tautology Theorem). *If $\mathcal{H} \vdash A_o^1$, ..., $\mathcal{H} \vdash A_o^n$ and $[A_o^1 \wedge \cdots \wedge A_o^n] \supset B_o$ is tautologous for $n \geq 1$, then $\mathcal{H} \vdash B_o$. Also, if B_o is tautologous, then $\mathcal{H} \vdash B_o$.*

Proof. Similar to the proof of theorem 5234 (Rule P) in [2]. See the comment above. ■

PROPOSITION 16. $\vdash [A_\alpha = B_\alpha] \supset [A_\alpha \simeq B_\alpha]$.

Proof. Follows from the definition of $\simeq$ and the Tautology Theorem. ■

THEOREM 17 (Deduction Theorem). *If $\mathcal{H} \cup \{H_o\} \vdash P_o$, then $\mathcal{H} \vdash H_o \supset P_o$.*

Proof. Similar to the proof of theorem 5240 in [2]. See the comment above. ■

7 Soundness and Completeness

In this section, let $\mathcal{H}$ be a set of wffs$_o$. $\mathcal{H}$ is *consistent* if there is no proof of F_o from $\mathcal{H}$ in $\mathcal{Q}_0^u$.

THEOREM 18 (Soundness Theorem). *If $\mathcal{H} \vdash \mathbf{A}_o$, then $\mathcal{H} \models \mathbf{A}_o$.*

Proof. A straightforward verification shows that (1) each instance of each axiom schema of $\mathcal{Q}_0^u$ is valid and (2) the rules of inference of $\mathcal{Q}_0^u$, R1 and R2, preserve validity in every general model for $\mathcal{Q}_0^u$. This shows that if $\vdash \mathbf{A}_o$, then $\models \mathbf{A}_o$.

Suppose $\mathcal{H} \vdash \mathbf{A}_o$ and $\mathcal{M}$ is a model for $\mathcal{H}$. Then there is a finite subset $\{\mathbf{H}_o^1, \ldots, \mathbf{H}_o^n\}$ of $\mathcal{H}$ such that $\{\mathbf{H}_o^1, \ldots, \mathbf{H}_o^n\} \vdash \mathbf{A}_o$. By the Deduction Theorem, this implies $\vdash \mathbf{H}_o^1 \supset \cdots \supset \mathbf{H}_o^n \supset \mathbf{A}_o$. By the result just above, $\mathcal{M} \models \mathbf{H}_o^1 \supset \cdots \supset \mathbf{H}_o^n \supset \mathbf{A}_o$. But $\mathcal{M} \models \mathbf{H}_o^i$ for all i with $1 \leq i \leq n$ since $\mathcal{M}$ is a model for $\mathcal{H}$. Therefore $\mathcal{M} \models \mathbf{A}_o$, and so $\mathcal{H} \models \mathbf{A}_o$. ∎

THEOREM 19 (Consistency Theorem). *If $\mathcal{H}$ has a general model, then $\mathcal{H}$ is consistent.*

Proof. Let $\mathcal{M}$ be a general model for $\mathcal{H}$. Assume that $\mathcal{H}$ is inconsistent, i.e., that $\mathcal{H} \vdash F_o$. Then, by the Soundness Theorem, $\mathcal{H} \models F_o$ and hence $\mathcal{M} \models F_o$. This means that $\mathcal{V}_\varphi^{\mathcal{M}}(F_o) = \mathsf{T}$ (for any assignment φ), which contradicts the definition of a general model. ∎

A *cwff* [*cwff$_\alpha$*] is a closed wff [closed wff$_\alpha$]. A *sentence* is a cwff$_o$. Let $\mathcal{H}$ be a set of sentences. $\mathcal{H}$ is *complete* in $\mathcal{Q}_0^u$ if, for every sentence $\mathbf{A}_o$, either $\mathcal{H} \vdash \mathbf{A}_o$ or $\mathcal{H} \vdash {\sim}\mathbf{A}_o$. $\mathcal{H}$ is *extensionally complete* in $\mathcal{Q}_0^u$ if, for every sentence of the form $\mathbf{A}_{\alpha\beta} = \mathbf{B}_{\alpha\beta}$, there is a cwff $\mathbf{C}_\beta$ such that:

1. $\mathcal{H} \vdash \mathbf{C}_\beta{\downarrow}$.

2. $\mathcal{H} \vdash [\mathbf{A}_{\alpha\beta}{\downarrow} \wedge \mathbf{B}_{\alpha\beta}{\downarrow} \wedge [\mathbf{A}_{\alpha\beta}\mathbf{C}_\beta \simeq \mathbf{B}_{\alpha\beta}\mathbf{C}_\beta]] \supset [\mathbf{A}_{\alpha\beta} = \mathbf{B}_{\alpha\beta}]$.

Let $\mathcal{L}(\mathcal{Q}_0^u)$ be the set of wffs of $\mathcal{Q}_0^u$.

LEMMA 20 (Extension Lemma). *Let $\mathcal{G}$ be a consistent set of sentences of $\mathcal{Q}_0^u$. Then there is an expansion $\overline{\mathcal{Q}_0^u}$ of $\mathcal{Q}_0^u$ and a set $\mathcal{H}$ of sentences of $\overline{\mathcal{Q}_0^u}$ such that:*

1. *$\mathcal{G} \subseteq \mathcal{H}$.*

2. *$\mathcal{H}$ is consistent.*

3. *$\mathcal{H}$ is complete in $\overline{\mathcal{Q}_0^u}$.*

4. $\mathcal{H}$ is extensionally complete in $\overline{\mathcal{Q}_0^{\mathrm{u}}}$.

5. $card(\mathcal{L}(\overline{\mathcal{Q}_0^{\mathrm{u}}})) = card(\mathcal{L}(\mathcal{Q}_0^{\mathrm{u}}))$.

Proof. The proof is very close to the proof of theorem 5500 in [2]. The crucial difference is that, in case (c) of the definition of $\mathcal{G}_{\tau+1}$,

$$\mathcal{G}_{\tau+1} = \mathcal{G}_\tau \cup \{\sim[\mathbf{A}_{\alpha\beta}\downarrow \wedge \mathbf{B}_{\alpha\beta}\downarrow \wedge [\mathbf{A}_{\alpha\beta}\mathbf{c}_\beta \simeq \mathbf{B}_{\alpha\beta}\mathbf{c}_\beta]]\}$$

where $\mathbf{c}_\beta$ is the first constant in $\mathcal{C}_\beta$ that does not occur in $\mathcal{G}_\tau$ or $\mathbf{A}_{\alpha\beta} = \mathbf{B}_{\alpha\beta}$. (Notice that $\vdash \mathbf{c}_\beta\downarrow$ by A6.)

To prove that $\mathcal{G}_{\tau+1}$ is consistent assuming $\mathcal{G}_\tau$ is consistent when $\mathcal{G}_{\tau+1}$ is obtained by case (c) , it is necessary to show that, if

$$\mathcal{G}_\tau \vdash \mathbf{A}_{\alpha\beta}\downarrow \wedge \mathbf{B}_{\alpha\beta}\downarrow \wedge [\mathbf{A}_{\alpha\beta}\mathbf{c}_\beta \simeq \mathbf{B}_{\alpha\beta}\mathbf{c}_\beta],$$

then $\mathcal{G}_\tau \vdash \mathbf{A}_{\alpha\beta} = \mathbf{B}_{\alpha\beta}$. Assume the hypothesis of this statement. Let $\mathcal{P}$ be a proof of

$$\mathbf{A}_{\alpha\beta}\downarrow \wedge \mathbf{B}_{\alpha\beta}\downarrow \wedge [\mathbf{A}_{\alpha\beta}\mathbf{c}_\beta \simeq \mathbf{B}_{\alpha\beta}\mathbf{c}_\beta]$$

from a finite subset $\mathcal{S}$ of $\mathcal{G}_\tau$, and let $\mathbf{x}_\beta$ be a variable that does not occur in $\mathcal{P}$ or $\mathcal{S}$. Since $\mathbf{c}_\beta$ does not occur in $\mathcal{G}_\tau$, $\mathbf{A}_{\alpha\beta}$, or $\mathbf{B}_{\alpha\beta}$, the result of substituting $\mathbf{x}_\beta$ for each occurrence of $\mathbf{c}_\beta$ in $\mathcal{P}$ is a proof of

$$\mathbf{A}_{\alpha\beta}\downarrow \wedge \mathbf{B}_{\alpha\beta}\downarrow \wedge [\mathbf{A}_{\alpha\beta}\mathbf{x}_\beta \simeq \mathbf{B}_{\alpha\beta}\mathbf{x}_\beta]$$

from $\mathcal{S}$. Therefore,

$$\mathcal{S} \vdash \mathbf{A}_{\alpha\beta}\downarrow \wedge \mathbf{B}_{\alpha\beta}\downarrow \wedge [\mathbf{A}_{\alpha\beta}\mathbf{x}_\beta \simeq \mathbf{B}_{\alpha\beta}\mathbf{x}_\beta].$$

This implies

$$\mathcal{S} \vdash \mathbf{A}_{\alpha\beta}\downarrow, \ \mathcal{S} \vdash \mathbf{B}_{\alpha\beta}\downarrow, \ \mathcal{S} \vdash \forall \mathbf{x}_\beta[\mathbf{A}_{\alpha\beta}\mathbf{x}_\beta \simeq \mathbf{B}_{\alpha\beta}\mathbf{x}_\beta]$$

by the Tautology Theorem and Universal Generalization since $\mathbf{x}_\beta$ does not occur in $\mathcal{S}$. It follows from these that $\mathcal{G}_\tau \vdash \mathbf{A}_{\alpha\beta} = \mathbf{B}_{\alpha\beta}$ by A3, Lemma 6, Proposition 16, Universal Generalization, Universal Instantiation, R1′, and R2′.

The rest of the proof is essentially the same as the proof of theorem 5500. ∎

A general model $\langle\{\mathcal{D}_\alpha \mid \alpha \in \mathcal{T}\}, \mathcal{J}\rangle$ for $\mathcal{Q}_0^{\mathrm{u}}$ is *frugal* if $card(\mathcal{D}_\alpha) \leq card(\mathcal{L}(\mathcal{Q}_0^{\mathrm{u}}))$ for all $\alpha \in \mathcal{T}$.

THEOREM 21 (Henkin's Theorem for $\mathcal{Q}_0^{\mathrm{u}}$). *Every consistent set of sentences of $\mathcal{Q}_0^{\mathrm{u}}$ has a frugal general model.*

Proof. Let $\mathcal{G}$ be a consistent set of sentences of $\mathcal{Q}_0^{\mathrm{u}}$, and let $\mathcal{H}$ and $\overline{\mathcal{Q}_0^{\mathrm{u}}}$ be as described in the Extension Lemma. We define simultaneously, by induction on $\gamma \in \mathcal{T}$, a frame $\{\mathcal{D}_\alpha \mid \alpha \in \mathcal{T}\}$ and a partial function $\mathcal{V}$ whose domain is the set of cwffs of $\overline{\mathcal{Q}_0^{\mathrm{u}}}$ so that the following conditions hold for all $\gamma \in \mathcal{T}$:

(1^γ) $\mathcal{D}_\gamma = \{\mathcal{V}(\mathbf{A}_\gamma) \mid \mathbf{A}_\gamma$ is a cwff$_\gamma$ and $\mathcal{H} \vdash \mathbf{A}_\gamma{\downarrow}\}$.

(2^γ) $\mathcal{V}(\mathbf{A}_\gamma)$ is defined iff $\mathcal{H} \vdash \mathbf{A}_\gamma{\downarrow}$ for all cwffs $\mathbf{A}_\gamma$.

(3^γ) $\mathcal{V}(\mathbf{A}_\gamma) = \mathcal{V}(\mathbf{B}_\gamma)$ iff $\mathcal{H} \vdash \mathbf{A}_\gamma = \mathbf{B}_\gamma$ for all cwffs $\mathbf{A}_\gamma$ and $\mathbf{B}_\gamma$.

Let $\mathcal{V}(x) \simeq \mathcal{V}(y)$ mean either $\mathcal{V}(x)$ and $\mathcal{V}(y)$ are both defined and equal or $\mathcal{V}(x)$ and $\mathcal{V}(y)$ are both undefined.

For each cwff $\mathbf{A}_o$, if $\mathcal{H} \vdash \mathbf{A}_o$, let $\mathcal{V}(\mathbf{A}_o) = \mathsf{T}$, and otherwise let $\mathcal{V}(\mathbf{A}_o) = \mathsf{F}$. Also, let $\mathcal{D}_o = \{\mathsf{T}, \mathsf{F}\}$. By the consistency and completeness of $\mathcal{H}$, exactly one of $\mathcal{H} \vdash \mathbf{A}_o$ and $\mathcal{H} \vdash {\sim}\mathbf{A}_o$ holds. Hence (1°) and (3°) are satisfied. (2°) is satisfied by Proposition 4.

For each cwff $\mathbf{A}_\imath$, if $\mathcal{H} \vdash \mathbf{A}_\imath{\downarrow}$, let

$$\mathcal{V}(\mathbf{A}_\imath) = \{\mathbf{B}_\imath \mid \mathbf{B}_\imath \text{ is a cwff}_\imath \text{ and } \mathcal{H} \vdash \mathbf{A}_\imath = \mathbf{B}_\imath\},$$

and otherwise let $\mathcal{V}(\mathbf{A}_\imath)$ be undefined. Also, let

$$\mathcal{D}_\imath = \{\mathcal{V}(\mathbf{A}_\imath) \mid \mathbf{A}_\imath \text{ is a cwff}_\imath \text{ and } \mathcal{H} \vdash \mathbf{A}_\imath{\downarrow}\}.$$

$(1^\imath)$, $(2^\imath)$, and $(3^\imath)$ are clearly satisfied.

Now suppose that $\mathcal{D}_\alpha$ and $\mathcal{D}_\beta$ are defined and that the conditions hold for α and β. For each cwff $\mathbf{A}_{\alpha\beta}$, if $\mathcal{H} \vdash \mathbf{A}_{\alpha\beta}{\downarrow}$, let $\mathcal{V}(\mathbf{A}_{\alpha\beta})$ be the (partial or total) function from $\mathcal{D}_\beta$ to $\mathcal{D}_\alpha$ whose value, for any argument $\mathcal{V}(\mathbf{B}_\beta) \in \mathcal{D}_\beta$, is $\mathcal{V}(\mathbf{A}_{\alpha\beta}\mathbf{B}_\beta)$ if $\mathcal{V}(\mathbf{A}_{\alpha\beta}\mathbf{B}_\beta)$ is defined and is undefined if $\mathcal{V}(\mathbf{A}_{\alpha\beta}\mathbf{B}_\beta)$ is undefined, and otherwise let $\mathcal{V}(\mathbf{A}_{\alpha\beta})$ be undefined. We must show that this definition is independent of the particular cwff $\mathbf{B}_\beta$ used to represent the argument. So suppose $\mathcal{V}(\mathbf{B}_\beta) = \mathcal{V}(\mathbf{C}_\beta)$; then $\mathcal{H} \vdash \mathbf{B}_\beta = \mathbf{C}_\beta$ by (3^β), so $\mathcal{H} \vdash \mathbf{A}_{\alpha\beta}\mathbf{B}_\beta \simeq \mathbf{A}_{\alpha\beta}\mathbf{C}_\beta$ by Lemmas 6 and 7 and R1′, and so $\mathcal{V}(\mathbf{A}_{\alpha\beta}\mathbf{B}_\beta) \simeq \mathcal{V}(\mathbf{A}_{\alpha\beta}\mathbf{C}_\beta)$ by (2^α) and (3^α), Finally, let

$$\mathcal{D}_{\alpha\beta} = \{\mathcal{V}(\mathbf{A}_{\alpha\beta}) \mid \mathbf{A}_{\alpha\beta} \text{ is a cwff}_{\alpha\beta} \text{ and } \mathcal{H} \vdash \mathbf{A}_{\alpha\beta}{\downarrow}\}.$$

$(1^{\alpha\beta})$ and $(2^{\alpha\beta})$ are clearly satisfied; we must show that $(3^{\alpha\beta})$ is satisfied. Suppose $\mathcal{V}(\mathbf{A}_{\alpha\beta}) = \mathcal{V}(\mathbf{B}_{\alpha\beta})$. Then $\mathcal{H} \vdash \mathbf{A}_{\alpha\beta}{\downarrow}$ and $\mathcal{H} \vdash \mathbf{B}_{\alpha\beta}{\downarrow}$. Since $\mathcal{H}$ is extensionally complete, there is a $\mathbf{C}_\beta$ such that $\mathcal{H} \vdash \mathbf{C}_\beta{\downarrow}$ and

$$\mathcal{H} \vdash [\mathbf{A}_{\alpha\beta}{\downarrow} \wedge \mathbf{B}_{\alpha\beta}{\downarrow} \wedge [\mathbf{A}_{\alpha\beta}\mathbf{C}_\beta \simeq \mathbf{B}_{\alpha\beta}\mathbf{C}_\beta]] \supset [\mathbf{A}_{\alpha\beta} = \mathbf{B}_{\alpha\beta}].$$

Then $\mathcal{V}(\mathbf{A}_{\alpha\beta}\mathbf{C}_\beta) \simeq \mathcal{V}(\mathbf{A}_{\alpha\beta})(\mathcal{V}(\mathbf{C}_\beta)) \simeq \mathcal{V}(\mathbf{B}_{\alpha\beta})(\mathcal{V}(\mathbf{C}_\beta)) \simeq \mathcal{V}(\mathbf{B}_{\alpha\beta}\mathbf{C}_\beta)$, so $\mathcal{H} \vdash \mathbf{A}_{\alpha\beta}\mathbf{C}_\beta \simeq \mathbf{B}_{\alpha\beta}\mathbf{C}_\beta$ by (2^α) and (3^α), and so $\mathcal{H} \vdash \mathbf{A}_{\alpha\beta} = \mathbf{B}_{\alpha\beta}$. Now suppose $\mathcal{H} \vdash \mathbf{A}_{\alpha\beta} = \mathbf{B}_{\alpha\beta}$. Then, for all cwffs $\mathbf{C}_\beta \in \mathcal{D}_\beta$, $\mathcal{H} \vdash \mathbf{A}_{\alpha\beta}\mathbf{C}_\beta \simeq \mathbf{B}_{\alpha\beta}\mathbf{C}_\beta$

by Lemmas 6 and 7 and R1$'$, and so $\mathcal{V}(\mathbf{A}_{\alpha\beta})(\mathcal{V}(\mathbf{C}_\beta)) \simeq \mathcal{V}(\mathbf{A}_{\alpha\beta}\mathbf{C}_\beta) \simeq \mathcal{V}(\mathbf{B}_{\alpha\beta}\mathbf{C}_\beta) \simeq \mathcal{V}(\mathbf{B}_{\alpha\beta})(\mathcal{V}(\mathbf{C}_\beta))$. Hence $\mathcal{V}(\mathbf{A}_{\alpha\beta}) = \mathcal{V}(\mathbf{B}_{\alpha\beta})$.

We claim that $\mathcal{M} = \langle\{\mathcal{D}_\alpha \mid \alpha \in \mathcal{T}\}, \mathcal{V}\rangle$ is an interpretation. For each primitive constant $\mathbf{c}_\gamma$ of $\overline{\mathcal{Q}_0^{\mathrm{u}}}$, $\mathcal{H} \vdash \mathbf{c}_\gamma$ by A6, and thus $\mathcal{V}$ maps each primitive constant of $\overline{\mathcal{Q}_0^{\mathrm{u}}}$ of type γ into $\mathcal{D}_\gamma$ by (1^γ) and (2^γ).

We must show that $\mathcal{V}(\mathsf{Q}_{o\alpha\alpha})$ is the identity relation on $\mathcal{D}_\alpha$. Let $\mathcal{V}(\mathbf{A}_\alpha)$ and $\mathcal{V}(\mathbf{B}_\alpha)$ be arbitrary members of $\mathcal{D}_\alpha$. Then $\mathcal{V}(\mathbf{A}_\alpha) = \mathcal{V}(\mathbf{B}_\alpha)$ iff $\mathcal{H} \vdash \mathbf{A}_\alpha = \mathbf{B}_\alpha$ iff $\mathcal{H} \vdash \mathsf{Q}_{o\alpha\alpha}\mathbf{A}_\alpha\mathbf{B}_\alpha$ iff $\mathsf{T} = \mathcal{V}(\mathsf{Q}_{o\alpha\alpha}\mathbf{A}_\alpha\mathbf{B}_\alpha) = \mathcal{V}(\mathsf{Q}_{o\alpha\alpha})(\mathcal{V}(\mathbf{A}_\alpha))(\mathcal{V}(\mathbf{B}_\alpha))$. Thus $\mathcal{V}(\mathsf{Q}_{o\alpha\alpha})$ is the identity relation on $\mathcal{D}_\alpha$.

We must show that, for $\alpha \neq o$, $\mathcal{V}(\iota_{\alpha(o\alpha)})$ is the unique member selector on $\mathcal{D}_\alpha$. For $\alpha \neq o$, let $\mathbf{A}_{o\alpha}$ be an arbitrary member of $\mathcal{D}_{o\alpha}$, $\mathbf{B}_\alpha$ be an arbitrary member of $\mathcal{D}_\alpha$, and $\mathbf{x}_\alpha$ be a variable that does not occur in $\mathbf{A}_{o\alpha}$. Using A12 and A13, $\mathcal{V}(\mathbf{A}_{o\alpha}) = \mathcal{V}(\mathsf{Q}_{o\alpha\alpha}\mathbf{B}_\alpha)$ iff $\mathcal{H} \vdash \mathbf{A}_{o\alpha} = \mathsf{Q}_{o\alpha\alpha}\mathbf{B}_\alpha$ iff $\mathcal{H} \vdash \iota_{\alpha(o\alpha)}\mathbf{A}_{o\alpha} = \mathbf{B}_\alpha$ iff $\mathcal{V}(\iota_{\alpha(o\alpha)}\mathbf{A}_{o\alpha}) = \mathcal{V}(\mathbf{B}_\alpha)$ iff $\mathcal{V}(\iota_{\alpha(o\alpha)})(\mathcal{V}(\mathbf{A}_{o\alpha})) = \mathcal{V}(\mathbf{B}_\alpha)$. Similarly, using A12 and A13, $\mathcal{V}(\sim\exists_1\mathbf{x}_\alpha[\mathbf{A}_{o\alpha}\mathbf{x}_\alpha]) = \mathsf{T}$ iff $\mathcal{H} \vdash \sim\exists_1\mathbf{x}_\alpha[\mathbf{A}_{o\alpha}\mathbf{x}_\alpha]$ iff $\mathcal{H} \vdash \iota_{\alpha(o\alpha)}\mathbf{A}_{o\alpha}\!\uparrow$ iff $\mathcal{V}(\iota_{\alpha(o\alpha)}\mathbf{A}_{o\alpha})$ is undefined iff $\mathcal{V}(\iota_{\alpha(o\alpha)})(\mathcal{V}(\mathbf{A}_{o\alpha}))$ is undefined. Thus $\mathcal{V}(\iota_{\alpha(o\alpha)})$ is the unique member selector on $\mathcal{D}_\alpha$.

Thus $\mathcal{M}$ is an interpretation. We claim further that $\mathcal{M}$ is a general model for $\overline{\mathcal{Q}_0^{\mathrm{u}}}$. For each assignment φ into $\mathcal{M}$ and wff $\mathbf{C}_\gamma$, let

$$\mathbf{C}_\gamma^\varphi = \mathsf{S}_{\mathbf{E}_{\delta_1}^1 \cdots \mathbf{E}_{\delta_n}^n}^{\mathbf{x}_{\delta_1}^1 \cdots \mathbf{x}_{\delta_n}^n} \mathbf{C}_\gamma$$

where $\mathbf{x}_{\delta_1}^1 \cdots \mathbf{x}_{\delta_n}^n$ are the free variables of $\mathbf{C}_\gamma$ and $\mathbf{E}_{\delta_i}^i$ is the first cwff (in some fixed enumeration) of $\overline{\mathcal{Q}_0^{\mathrm{u}}}$ such that $\varphi(\mathbf{x}_{\delta_i}^i) = \mathcal{V}(\mathbf{E}_{\delta_i}^i)$ for all i with $1 \leq i \leq n$.[6] Let $\mathcal{V}_\varphi(\mathbf{C}_\gamma) \simeq \mathcal{V}(\mathbf{C}_\gamma^\varphi)$. $\mathbf{C}_\gamma^\varphi$ is clearly a cwff$_\gamma$, so $\mathcal{V}_\varphi(\mathbf{C}_\gamma) \in \mathcal{D}_\gamma$ if $\mathcal{V}_\varphi(\mathbf{C}_\gamma)$ is defined.

(a) Let $\mathbf{C}_\gamma$ be a variable $\mathbf{x}_\delta$. Choose $\mathbf{E}_\delta$ so that $\varphi(\mathbf{x}_\delta) = \mathcal{V}(\mathbf{E}_\delta)$ as above. Then $\mathcal{V}_\varphi(\mathbf{C}_\gamma) = \mathcal{V}_\varphi(\mathbf{x}_\delta) = \mathcal{V}(\mathbf{x}_\delta^\varphi) = \mathcal{V}(\mathbf{E}_\delta) = \varphi(\mathbf{x}_\delta)$.

(b) Let $\mathbf{C}_\gamma$ be a primitive constant. Then $\mathcal{V}_\varphi(\mathbf{C}_\gamma) = \mathcal{V}(\mathbf{C}_\gamma^\varphi) = \mathcal{V}(\mathbf{C}_\gamma)$.

(c) Let $\mathbf{C}_\gamma$ be $[\mathbf{A}_{\alpha\beta}\mathbf{B}_\beta]$. If $\mathcal{V}_\varphi(\mathbf{A}_{\alpha\beta})$ is defined, $\mathcal{V}_\varphi(\mathbf{B}_\beta)$ is defined, and $\mathcal{V}_\varphi(\mathbf{A}_{\alpha\beta})$ is defined at $\mathcal{V}_\varphi(\mathbf{B}_\beta)$, then $\mathcal{V}_\varphi(\mathbf{C}_\gamma) = \mathcal{V}_\varphi(\mathbf{A}_{\alpha\beta}\mathbf{B}_\beta) = \mathcal{V}(\mathbf{A}_{\alpha\beta}^\varphi\mathbf{B}_\beta^\varphi) = \mathcal{V}(\mathbf{A}_{\alpha\beta}^\varphi)(\mathcal{V}(\mathbf{B}_\beta^\varphi)) = \mathcal{V}_\varphi(\mathbf{A}_{\alpha\beta})(\mathcal{V}_\varphi(\mathbf{B}_\beta))$. Now assume $\mathcal{V}_\varphi(\mathbf{A}_{\alpha\beta})$ is undefined, $\mathcal{V}_\varphi(\mathbf{B}_\beta)$ is undefined, or $\mathcal{V}_\varphi(\mathbf{A}_{\alpha\beta})$ is not defined at $\mathcal{V}_\varphi(\mathbf{B}_\beta)$. Then $\mathcal{H} \vdash \mathbf{A}_{\alpha\beta}^\varphi\!\uparrow$, $\mathcal{H} \vdash \mathbf{B}_\beta^\varphi\!\uparrow$, or $\mathcal{V}(\mathbf{A}_{\alpha\beta}^\varphi\mathbf{B}_\beta^\varphi)$ is undefined. If $\alpha = o$, then $\mathcal{H} \vdash \mathbf{A}_{\alpha\beta}^\varphi\!\uparrow$ or $\mathcal{H} \vdash \mathbf{B}_\beta^\varphi\!\uparrow$, which implies $\mathcal{H} \vdash \sim\mathbf{A}_{\alpha\beta}^\varphi\mathbf{B}_\beta^\varphi$ by A9, so $\mathcal{H} \vdash \mathbf{A}_{\alpha\beta}^\varphi\mathbf{B}_\beta^\varphi = F_o$, so $\mathcal{V}(\mathbf{A}_{\alpha\beta}^\varphi\mathbf{B}_\beta^\varphi) = \mathcal{V}(F_o)$,

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[6] $\mathsf{S}_{\mathbf{E}_{\delta_1}^1 \cdots \mathbf{E}_{\delta_n}^n}^{\mathbf{x}_{\delta_1}^1 \cdots \mathbf{x}_{\delta_n}^n} \mathbf{C}_\gamma$ is the result of simultaneously substituting $\mathbf{E}_{\delta_i}^i$ for each free occurrence of $\mathbf{x}_{\delta_i}^i$ in $\mathbf{C}_\gamma$ for all i with $1 \leq i \leq n$.

so $\mathcal{V}_\varphi(\mathbf{A}_{\alpha\beta}\mathbf{B}_\beta) = \mathcal{V}(F_o)$, and so $\mathcal{V}_\varphi(\mathbf{C}_\gamma) = \mathsf{F}$. If $\alpha \neq o$ and $\mathcal{H} \vdash \mathbf{A}_{\alpha\beta}^\varphi{\uparrow}$ or $\mathcal{H} \vdash \mathbf{B}_\beta^\varphi{\uparrow}$, then $\mathcal{H} \vdash [\mathbf{A}_{\alpha\beta}^\varphi\mathbf{B}_\beta^\varphi]{\uparrow}$ by A10, and so $\mathcal{V}(\mathbf{A}_{\alpha\beta}^\varphi\mathbf{B}_\beta^\varphi)$ is undefined. Hence, if $\alpha \neq o$, $\mathcal{V}_\varphi(\mathbf{C}_\gamma) \simeq \mathcal{V}_\varphi(\mathbf{A}_{\alpha\beta}\mathbf{B}_\beta) \simeq \mathcal{V}(\mathbf{A}_{\alpha\beta}^\varphi\mathbf{B}_\beta^\varphi)$ is undefined.

(d) Let $\mathbf{C}_\gamma$ be $[\lambda\mathbf{x}_\alpha\mathbf{B}_\beta]$. Let $\mathcal{V}(\mathbf{E}_\alpha)$ be an arbitrary member of $\mathcal{D}_\alpha$, and so $\mathbf{E}_\alpha$ is a cwff and $\mathcal{H} \vdash \mathbf{E}_\alpha{\downarrow}$. Given an assignment φ, let $\psi = (\varphi : \mathbf{x}_\alpha/\mathcal{V}(\mathbf{E}_\alpha))$. From A4 it follows that $\mathcal{H} \vdash [\lambda\mathbf{x}_\alpha\mathbf{B}_\beta]^\varphi\mathbf{E}_\alpha \simeq \mathbf{B}_\beta^\psi$. Then $\mathcal{V}_\varphi(\mathbf{C}_\gamma)(\mathcal{V}(\mathbf{E}_\alpha)) \simeq \mathcal{V}([\lambda\mathbf{x}_\alpha\mathbf{B}_\beta]^\varphi)(\mathcal{V}(\mathbf{E}_\alpha)) \simeq \mathcal{V}(\mathbf{B}_\beta^\psi) \simeq \mathcal{V}_\psi(\mathbf{B}_\beta)$. Thus $\mathcal{V}_\varphi(\mathbf{C}_\gamma)$ satisfies condition (d) in the definition of a general model.

Thus $\mathcal{M}$ is a general model for $\overline{\mathcal{Q}_0^{\mathrm{u}}}$ (and hence for $\mathcal{Q}_0^{\mathrm{u}}$). Also, if $\mathbf{A}_o \in \mathcal{G}$, then $\mathbf{A}_o \in \mathcal{H}$, so $\mathcal{H} \vdash \mathbf{A}_o$, so $\mathcal{V}(\mathbf{A}_o) = \mathsf{T}$ and $\mathcal{M} \models \mathbf{A}_o$, so $\mathcal{M}$ is a general model for $\mathcal{G}$. Clearly, (1) $\mathrm{card}(\mathcal{D}_\alpha) \leq \mathrm{card}(\mathcal{L}(\mathcal{Q}_0^{\mathrm{u}}))$ since $\mathcal{V}$ maps a subset of the cwffs$_\alpha$ of $\overline{\mathcal{Q}_0^{\mathrm{u}}}$ onto $\mathcal{D}_\alpha$ and (2) $\mathrm{card}(\mathcal{L}(\overline{\mathcal{Q}_0^{\mathrm{u}}})) = \mathrm{card}(\mathcal{L}(\mathcal{Q}_0^{\mathrm{u}}))$, and so $\mathcal{M}$ is frugal. ∎

THEOREM 22 (Henkin's Completeness Theorem for $\mathcal{Q}_0^{\mathrm{u}}$). *Let $\mathcal{H}$ be a set of sentences of $\mathcal{Q}_0^{\mathrm{u}}$. If $\mathcal{H} \models \mathbf{A}_o$, then $\mathcal{H} \vdash \mathbf{A}_o$.*

Proof. Assume $\mathcal{H} \models \mathbf{A}_o$, and let $\mathbf{B}_o$ be the universal closure of $\mathbf{A}_o$. Then $\mathcal{H} \models \mathbf{B}_o$. Suppose $\mathcal{H} \cup \{{\sim}\mathbf{B}_o\}$ is consistent. Then, by Henkin's Theorem, there is a general model $\mathcal{M}$ for $\mathcal{H} \cup \{{\sim}\mathbf{B}_o\}$, and so $\mathcal{M} \models {\sim}\mathbf{B}_o$. Since $\mathcal{M}$ is also a general model for $\mathcal{H}$, $\mathcal{M} \models \mathbf{B}_o$. From this contradiction it follows that $\mathcal{H} \cup \{{\sim}\mathbf{B}_o\}$ is inconsistent. Hence $\mathcal{H} \vdash \mathbf{B}_o$ by the Deduction Theorem and the Tautology Theorem. Therefore, $\mathcal{H} \vdash \mathbf{A}_o$ by Universal Instantiation and A5. ∎

8   Conclusion

$\mathcal{Q}_0^{\mathrm{u}}$ is a version of Church's type theory that directly formalizes the traditional approach to undefinedness. In this paper we have presented the syntax, semantics, and proof system of $\mathcal{Q}_0^{\mathrm{u}}$. The semantics is based on Henkin-style general models. We have also proved that $\mathcal{Q}_0^{\mathrm{u}}$ is sound and complete with respect to its semantics.

$\mathcal{Q}_0^{\mathrm{u}}$ is a modification of $\mathcal{Q}_0$. Its syntax is essentially identical to the syntax of $\mathcal{Q}_0$. Its semantics is based on general models that include partial functions as well as total functions and in which terms may be nondenoting. Its proof system is derived from the proof system of $\mathcal{Q}_0$; the axiom schemas and rules of inference of $\mathcal{Q}_0$ have been modified to accommodate partial functions and undefined terms and to axiomatize definite description.

Our presentation of $\mathcal{Q}_0^{\mathrm{u}}$ is intended to show as clearly as possible what must be changed in Church's type theory in order to formalize the traditional approach to undefinedness. Our development of $\mathcal{Q}_0^{\mathrm{u}}$ closely follows

Andrews' development of $\mathcal{Q}_0$. Notes indicate where and how $\mathcal{Q}_0$ and $\mathcal{Q}_0^{\mathrm{u}}$ differ from each other. And the proofs of the soundness and completeness theorems for $\mathcal{Q}_0^{\mathrm{u}}$ follow very closely the proofs of these theorems for $\mathcal{Q}_0$.

$\mathcal{Q}_0$ and $\mathcal{Q}_0^{\mathrm{u}}$ have the same *theoretical expressivity* (see Note 6). However, with its formalization of the traditional approach, $\mathcal{Q}_0^{\mathrm{u}}$ has significantly greater *practical expressivity* than $\mathcal{Q}_0$. Statements involving partial functions and undefined terms can be expressed in $\mathcal{Q}_0^{\mathrm{u}}$ more naturally and concisely than in $\mathcal{Q}_0$ (see [9]). All the standard laws of predicate logic hold in $\mathcal{Q}_0^{\mathrm{u}}$ except those involving equality and substitution, but these do hold for defined terms. In summary, $\mathcal{Q}_0^{\mathrm{u}}$ has the benefit of greater practical expressivity at the cost of a modest departure from standard predicate logic.

The benefits of a practical logic like $\mathcal{Q}_0^{\mathrm{u}}$ would be best realized by a computer implementation of the logic. $\mathcal{Q}_0^{\mathrm{u}}$ has not been implemented, but the related logic LUTINS [6, 7, 8] has been implemented in the IMPS theorem proving system [10, 11] and successfully used to prove hundreds of theorems in traditional mathematics, especially in mathematical analysis. LUTINS is essentially just a more sophisticated version of $\mathcal{Q}_0^{\mathrm{u}}$ with subtypes and additional expression constructors. An implemented logic that formalizes the traditional approach to undefinedness can reap the benefits of a proven approach developed in mathematical practice over hundreds of years.

9   Acknowledgments

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